Efficient Algebraic Coarse Spaces

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Outline

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Linear Systems

Three methods exist for solving linear systems:

- Direct methods: LU, Cholesky, LDL^T
- Iterative methods: Gauss-Seidel, Jacobi, Krylov methods
- Hybrid methods: **Domain decomposition**, Multigrid

How to construct algebraic preconditioners such that $\kappa_2(M^{-1}A) \leq a$?

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How to construct algebraic preconditioners such that $\kappa_2(M^{-1}A) \leq a$?

Algebraic Formulation of Additive Schwarz

$A \in \mathbb{R}^{n \times n}$ SPD. N non-overlapping subdomains of graph of A.

- $R_{i,0}$ restriction matrix to subdomain i
- $R_{i,\delta}$ restriction matrix to the boundary of subdomain i
- R_i restriction matrix $[R_{i,0}; R_{i,\delta}]$
- D_i partition of unity: $\sum_{i=1}^{N} R_i^\top D_i R_i = I_n$

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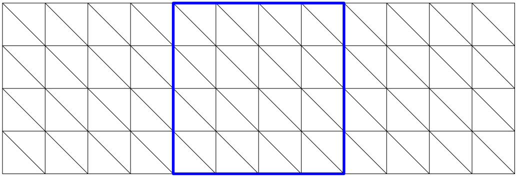

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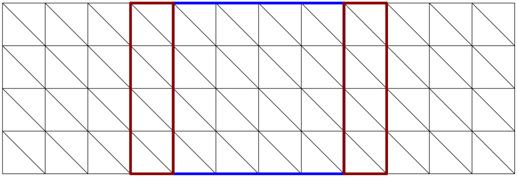

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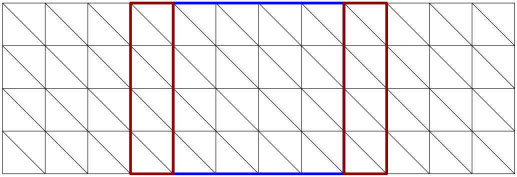

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The additive Schwarz preconditioner is defined as:

$$M_{AS}^{-1} = \sum_{i=1}^{N} R_i^\top A_i^{-1} R_i,$$

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- $\lambda_{max}(M_{AS}^{-1} A) \leq k_c$
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Number of Subdomains

Two-level Algebraic Additive Schwarz

$$M_{AS}^{-1} = R_0^\top A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^\top A_i^{-1} R_i,$$

where $A_i = R_i A R_i^\top$, $i = 0, 1, \ldots, N$.

For any Coarse Space $\mathcal{S}$

$$u^\top A u \leq (k_c + 1) u^\top M_{AS2} u.$$

Correction

If we have M_1 a block diagonal matrix such that

$$c_1 u^\top M_1 u \leq u^\top A u$$

Correct blocks of M_{AS} w.r.t. blocks of M_1 to have

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Local SPSD splitting

Using a permutation P_i, A can be written as:

$$A = P_i \begin{pmatrix} R_{i,0}AR_{i,0}^\top & R_{i,0}AR_{i,\delta}^\top & \\ R_{i,\delta}AR_{i,0}^\top & R_{i,\delta}AR_{i,\delta}^\top & R_{i,\delta}AR_{i,c}^\top \\ & R_{i,c}AR_{i,\delta}^\top & R_{i,c}AR_{i,c}^\top \end{pmatrix} P_i^\top$$

$\tilde{A}_i$ is SPSD splitting of A if

- $0 \le u^\top \tilde{A}_i u \le u^\top A u$
- $\tilde{A}_i = P_i \begin{pmatrix} R_{i,0}AR_{i,0}^\top & R_{i,0}AR_{i,\delta}^\top & \\ R_{i,\delta}AR_{i,0}^\top & A_{i,\delta} & \\ & & \end{pmatrix} P_i^\top$

Characterization

Lemma ([Al Daas and Grigori, 2018])

Equivalence

- $\tilde{A}_i = P_i \begin{pmatrix} R_{i,0}AR_{i,0}^\top & R_{i,0}AR_{i,\delta}^\top \\ R_{i,\delta}AR_{i,0}^\top & A_{i,\delta} \end{pmatrix} P_i^\top$ *is a SPSD splitting of A*
- $A_{i,\delta}$ *verifies:*

$$u^\top \overbrace{R_{i,\delta}AR_{i,0}^\top(R_{i,0}AR_{i,0}^\top)^{-1}R_{i,0}AR_{i,\delta}^\top}^{M_{LB}^i} u \leq u^\top A_{i,\delta} u$$

$$u^\top A_{i,\delta} u \leq u^\top \underbrace{\left(R_{i,\delta}AR_{i,\delta}^\top - R_{i,\delta}AR_{i,C}^\top(R_{i,C}AR_{i,C}^\top)^{-1}R_{i,C}AR_{i,\delta}^\top\right)}_{M_{UB}^i} u$$

Examples of SPSD Splittings

$A_{i,\delta}$ can be:

- M_{UB}^i (The Upper Bound)
- M_{LB}^i (The Lower Bound)
- any convex linear combination

Analytic choice (GenEO [Spillane et al., 2014, Dolean et al., 2015]): if PDE discretization by FEM, $A_{i,\delta}$ = integration over the elements related to $R_{i,\delta}$ (boundary of subdomain)

Local Filtering Subspaces

Definition [Al Daas and Grigori, 2018]

- $\tilde{A}_i$ SPSD splitting, $\tau > 0$
- $\tilde{Z}_i$ is a subspace
- $\tilde{P}_i$ is an orthogonal projection on $\tilde{Z}_i$

$\tilde{Z}_i$ is a τ-filtering subspace if

$$u_i^\top \left(R_i A R_i^\top\right) u_i \leq \tau \left(R_i u\right)^\top \left(R_i \tilde{A}_i R_i^\top\right) \left(R_i u\right), \quad \forall u \in \mathbb{R}^n,$$

where $u_i = \left(D_i \left(I - \tilde{P}_i\right) R_i u\right)$ and D_i is the partition of unity, for $i = 1, \ldots, N$.

Minimal Filtering Subspace

How to constuct τ-filtering subspaces?

Lemma ([Al Daas and Grigori, 2018])

$\tilde{A}_i$ local SPSD splittings of A, D_i is the partition of unity, $\tilde{G}_i = D_i \left(R_i A R_i^\top \right) D_i$. $\tau > 0$. If $\tilde{G}_i$ is definite, consider problem

$$R_i \tilde{A}_i R_i^\top u_{i,k} = \lambda_{i,k} \tilde{G}_i u_{i,k}.$$

Set $\tilde{Z}_{\tau,i} = span\left\{ u_{i,k} \mid \lambda_{i,k} < \frac{1}{\tau} \right\}$. Then, $\tilde{Z}_{\tau,i}$ is the smallest dimension τ-filtering subspace

The Coarse Space

Definition (Coarse space based on algebraic local SPSD splitting of A, ALS)

Let $\tilde{A}_i$, D_i, and $\tilde{Z}_{\tau,i}$ be defined as previously. We define $\mathcal{S}$ as

$$\mathcal{S} = \bigoplus_{i=1}^{N} R_i^\top D_i \tilde{Z}_{\tau,i}.$$

Let $\tilde{Z}_0$ be a matrix whose columns form a basis of $\mathcal{S}$. We denote its transpose by $R_0 = \tilde{Z}_0^\top$.

Bound on The Condition Number

Theorem ([Al Daas and Grigori, 2018])

Let M_{ALS} be the two-level AS preconditioner combined with the previous coarse space. The following inequality holds,

$$\kappa\left(M_{ALS}^{-1}A\right) \leq (k_c+1)\left(2+(2k_c+1)k_m\tau\right)$$

where k_c is the maximum number of neighbours and $k_m \leq N$ verifying

$$0 \leq \sum_{i=1}^{N} u^\top \tilde{A}_i u \leq k_m u^\top A u \quad \forall u \in \mathbb{R}^n,$$

Minimality of Filtering Subspace

$u^\top \tilde{A}_i^1 u \leq u^\top \tilde{A}_i^2 u$ local SPSD splittings. Z_1 and Z_2 are the corresponding minimal τ-filtering subspace.

$$u_i^\top \left(R_i A R_i^\top\right) u_i \leq \tau \, (R_i u)^\top \left(R_i \tilde{A}_i^1 R_i^\top\right)(R_i u), \quad \forall u \in \mathbb{R}^n,$$

then,

$$Z_2 \subset Z_1.$$

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Extreme Cases of The Local SPSD Splitting

1. M_{UB}^{i} (The Upper Bound) $\rightarrow$ the lowest dimension $\mathcal{S}$
2. M_{LB}^{i} (The Lower Bound)a $\rightarrow$ the largest dimension $\mathcal{S}$

Approximation of The Upper Bound

- $M^i_{UB} = R_{i,\delta}AR^\top_{i,\delta} - R_{i,\delta}AR^\top_{i,C}(R_{i,C}AR^\top_{i,C})^{-1}R_{i,C}AR^\top_{i,\delta}$ (upper bound)
- $\tilde{M}^i_{UB} = R_{i,\delta}AR^\top_{i,\delta} - R_{i,\delta}AR^\top_{i,\tilde{C}}(R_{i,\tilde{C}}AR^\top_{i,\tilde{C}})^{-1}R_{i,\tilde{C}}AR^\top_{i,\delta}$ ($\approx$ upper bound)

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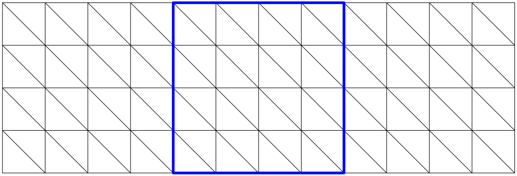

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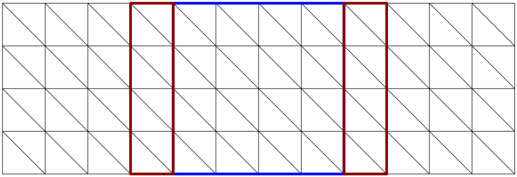

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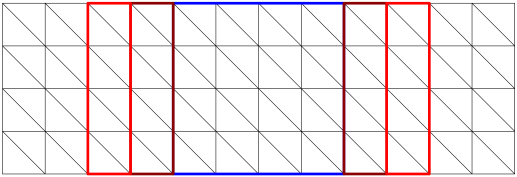

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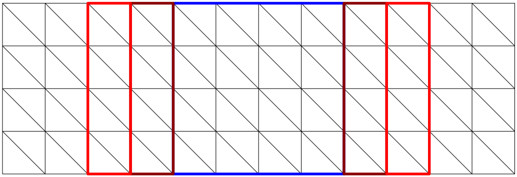

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Coarse Spaces minimality: Algebraic vs Analytic

Heterogeneous elasticity 3-D prob. $\kappa_2 \approx 10^{12}$

N	dim_{uC}	$n_{M_{UB}}$	dim_{Gen}	n_{Gen}	n_{AS}
4	82	20	106	20	-
8	179	23	229	24	-
16	304	37	391	38	-
32	447	53	614	42	-
64	622	84	850	55	-
128	969	131	1326	61	-

Approximation Algebraic Coarse Space

Model Reduction Problem (Suite Sparse Matrix Collection)

Matrix	n	N	$n_{M_{UB}}$	n_{app}	n_{AS}
		4	15	20	118
		8	16	22	480
LF10000	19998	16	18	25	-
		32	19	27	-
		64	20	33	-
		128	23	43	-

Outline

Summary and Perspectives

Summary

- How to split algebraically a SPD matrix
- Defined τ-filtering subspaces for constructing local coarse spaces
- Suggested an affordable approximation of robust algebraic coarse spaces

Perspectives

- study other approaches for approximation, e.g., connectivity in AMG
- relation with Optimized Schwarz methods
- 3-level Additive Schwarz

Thank you for your attention!

Questions?

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