

Design and Implementation of a Parallel Threshold Markowitz Algorithm

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Introduction

Goal: solve highly unsymmetric sparse systems.

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 - ▶ chemical engineering
 - ▶ linear programming
 - ▶ economic modelling
 - ▶ power systems

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 - ▶ linear programming
 - ▶ economic modelling
 - ▶ power systems

Both code and matrices can be very **evil**

Highly unsymmetric systems

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$$si(A) = \frac{\text{number}_{i \neq j} \{a_{ij} * a_{ji} \neq 0\}}{nz\{A\}}$$

Highly unsymmetric systems

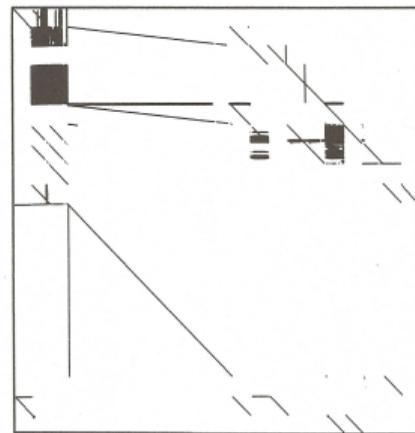
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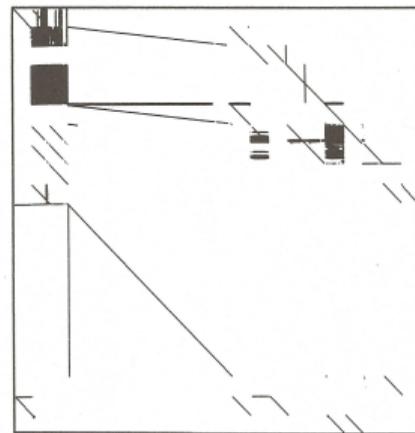


Matrix from econometric model of SE Asia

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Matrix from econometric model of SE Asia

State of the art solvers: MA48, UMFPACK, SuperLU, MUMPS.

Threshold Markowitz pivoting

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- ▶ Threshold value
Consider only entries a_{ij} that satisfy

$$|a_{ij}| \geq u * \max_k |a_{kj}|, k = 1, n$$

where u is a threshold parameter $0 < u \leq 1.0$.

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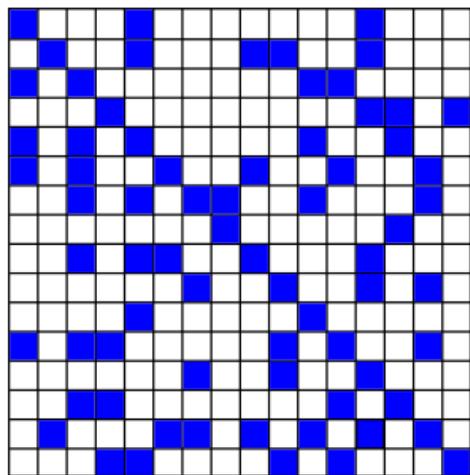
- ▶ Markowitz threshold

If there are r_i entries in row i and c_j entries in column j , the Markowitz count for the entry a_{ij} is given by $M(a_{ij}) = (r_i - 1)(c_j - 1)$. In each column, consider only the entries that satisfy the threshold test $M(a_{ij}) \leq \alpha * \min_{\text{markowitz}}$ where $\min_{\text{markowitz}}$ is the minimum Markowitz cost for an entry in the matrix.

High-level algorithm description

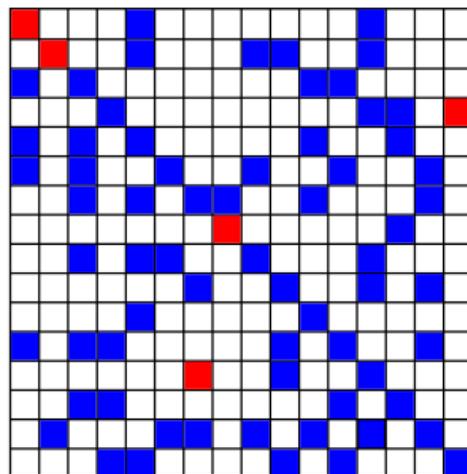
while $size(A) > 1$ **do**

end while



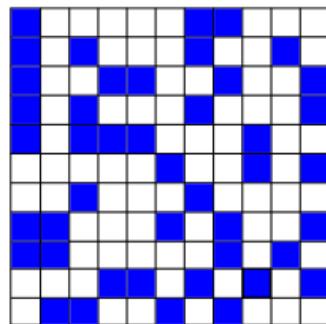
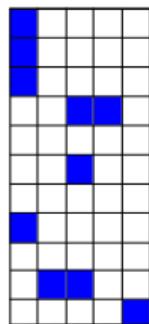
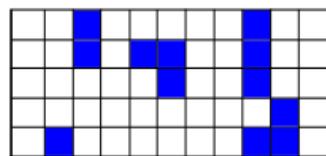
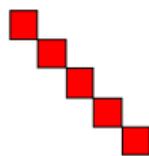
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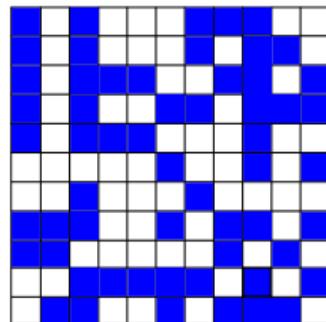
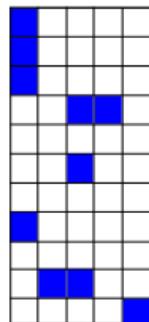
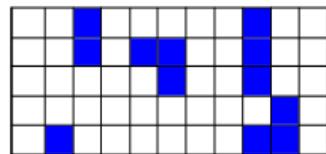
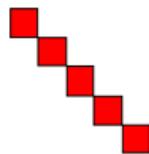
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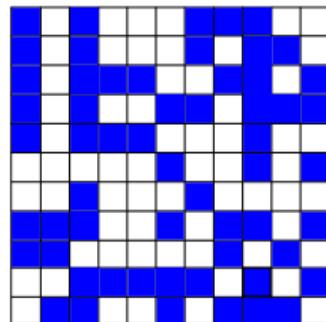
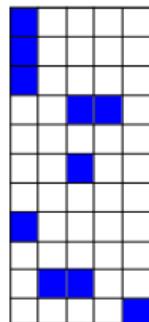
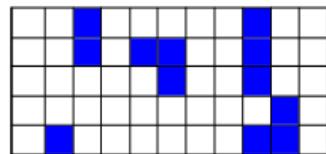
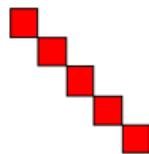
Update the trailing matrix

end while



High-level algorithm description

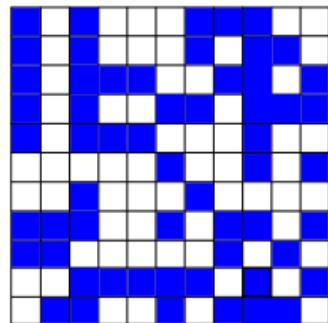
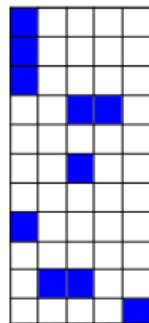
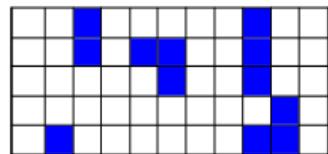
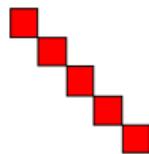
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while  $\text{density}(A) < \text{eps}$  do  
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High-level algorithm description

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Switch to dense factorization.



Luby's Algorithm for a **Maximal Independent Set (MIS)**

Input $G = (V, E)$ an undirected graph

Output $I \subseteq V$, an MIS

$I \leftarrow \emptyset$

$G' = (V', E') \leftarrow G = (V, E)$

while $V' \neq \emptyset$ **do**

assign random score to each node in V'

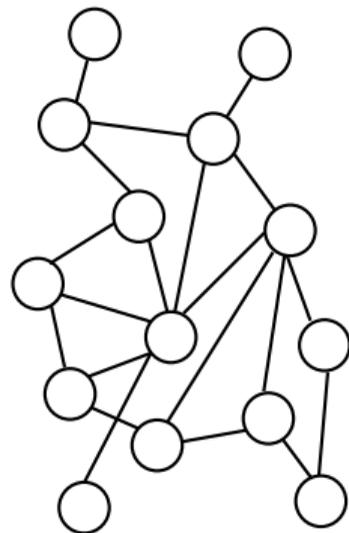
$I' \leftarrow$ nodes having highest score among their neighbours

$I \leftarrow I \cup I'$

$Y \leftarrow I' \cup N(I')$

Set $G' = (V', E')$ to the induced subgraph on $V' - Y$

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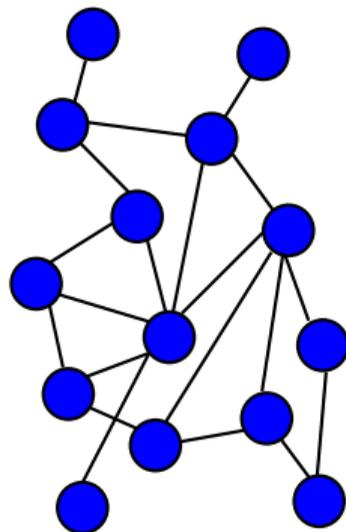
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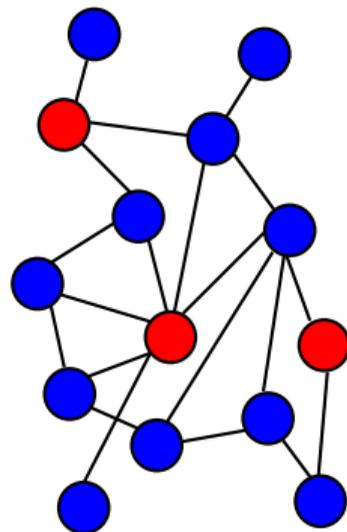
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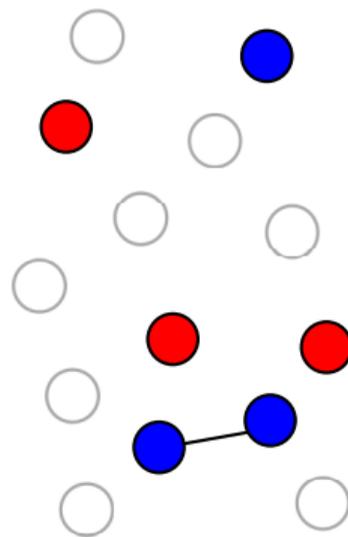
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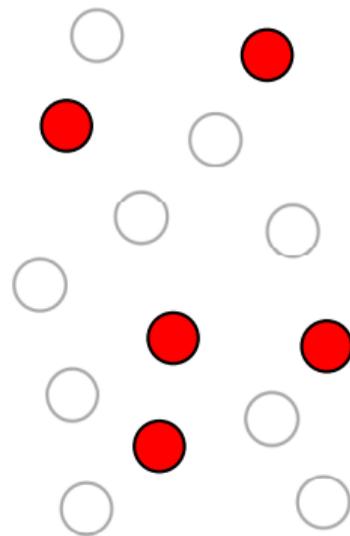
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We then continue the process to obtain an MIS with 5 nodes.

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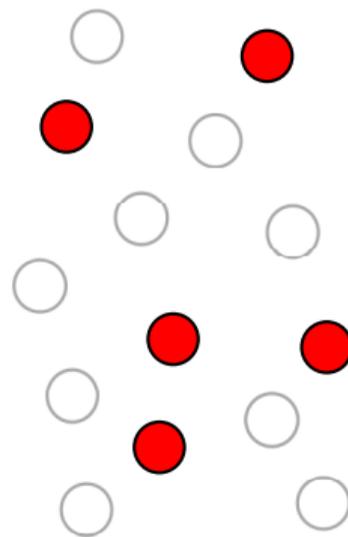
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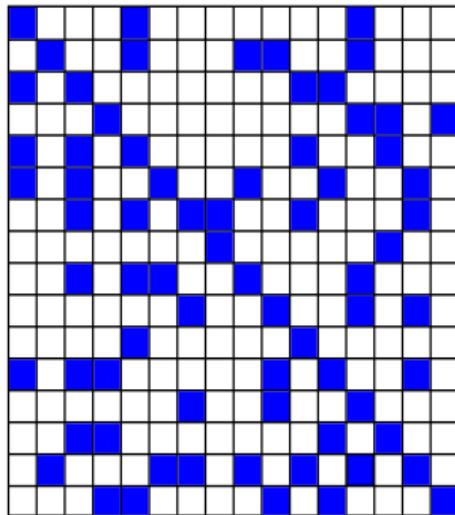
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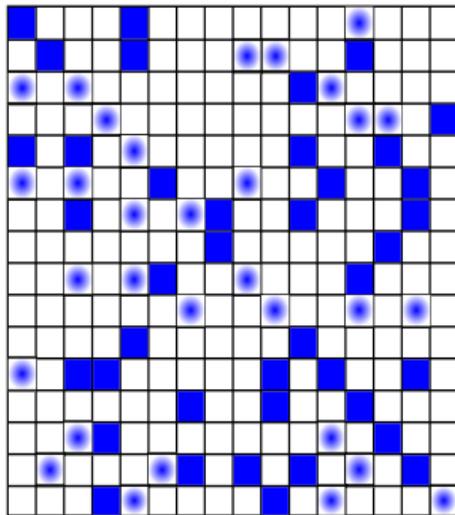
We have adopted the Luby's algorithm for directed graphs.

Parallel Solver for Highly Unsymmetric Matrices (ParSHUM) library



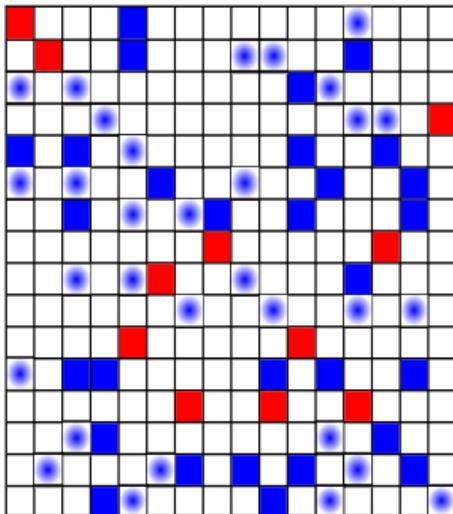
Parallel Solver for Highly Unsymmetric Matrices (ParSHUM) library

Discard all numerically ineligible entries and calculate the minimum Markowitz cost.



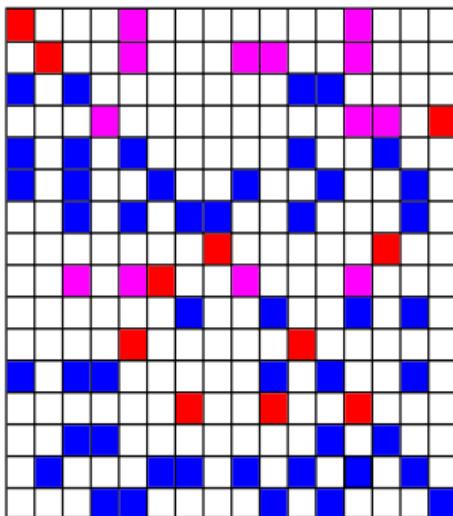
Parallel Solver for Highly Unsymmetric Matrices (ParSHUM) library

Choose potential pivots that satisfy the Markowitz threshold test, one for each column and assign a score for each one. The pivot score is associated with the column.



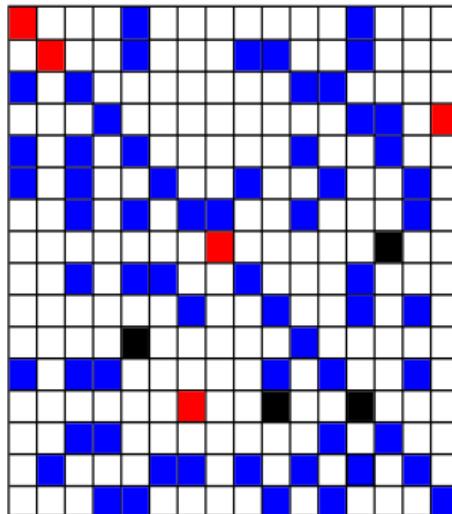
Parallel Solver for Highly Unsymmetric Matrices (ParSHUM) library

For each potential pivot scan its row, comparing its score with the score of the columns with entries in the row.



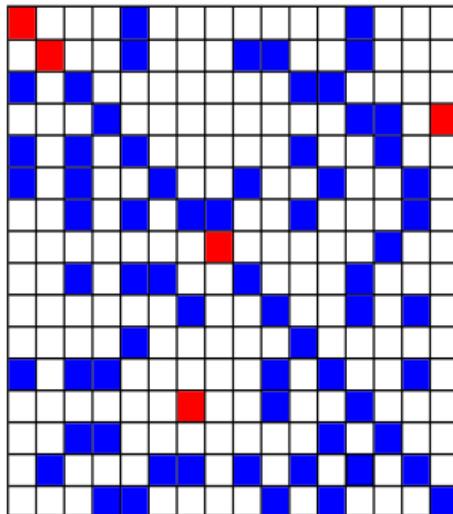
Parallel Solver for Highly Unsymmetric Matrices (ParSHUM) library

Discard the column with a potential pivot with lower score.



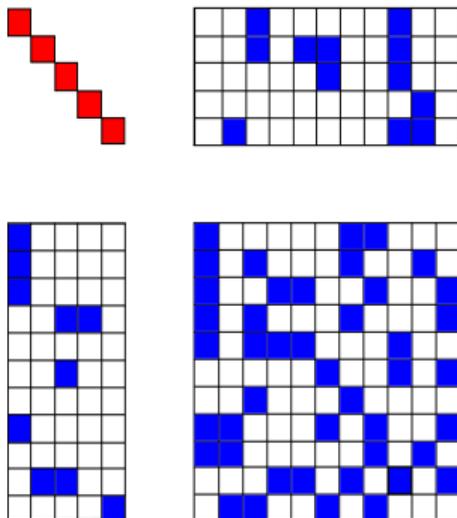
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The set of independent pivots.



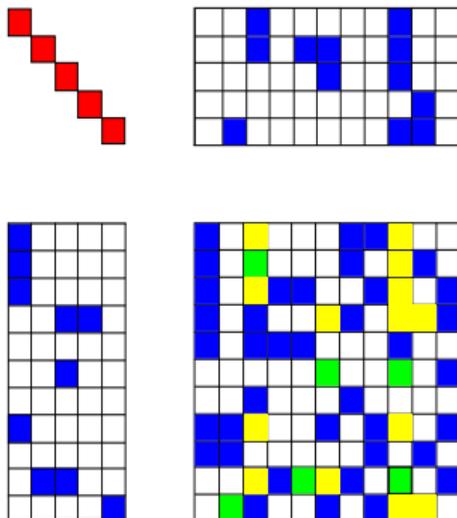
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This results in a reordered matrix of the form:



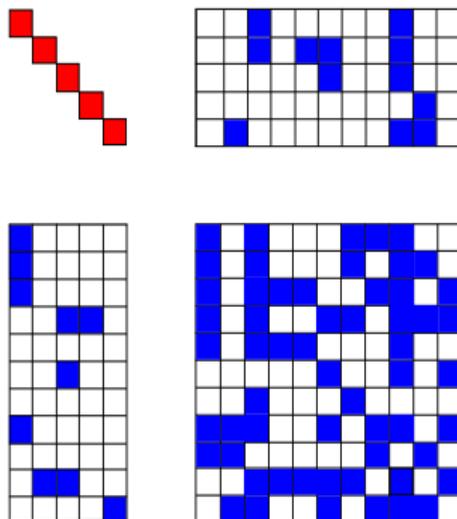
Parallel Solver for Highly Unsymmetric Matrices (ParSHUM) library

The trailing matrix is then updated by a sparse matrix-matrix multiply.



Parallel Solver for Highly Unsymmetric Matrices (ParSHUM) library

The pivot search is then repeated on the reduced matrix.



Experimental set-up

HPC2N platform at
Umeå University

Each node:

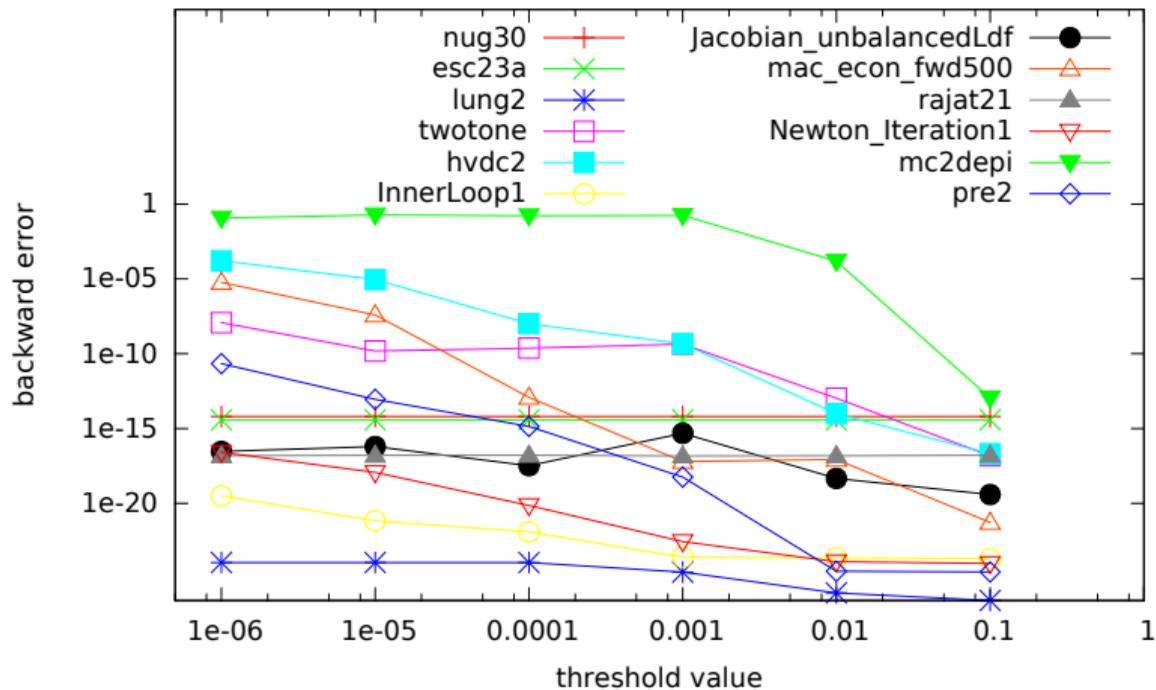
28 Intel Xeon E5-2690v4

35 MB of shared L3 cache

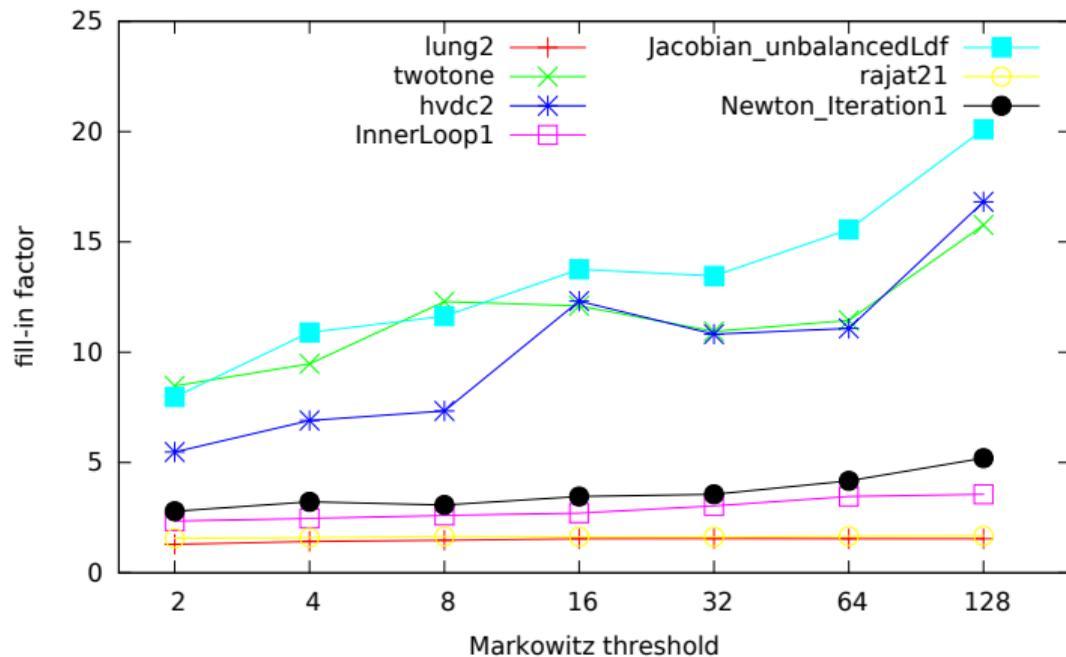
128 GB RAM memory

| Matrix | Order $\times 10^3$ | Entries $\times 10^6$ | <i>si</i> |
|------------------------|------------------------|--------------------------|-----------|
| nug30 | 52.4 | 0.24 | 0.00 |
| esc32a | 63.6 | 0.31 | 0.00 |
| lung2 | 109 | 0.49 | 0.57 |
| twotone | 120 | 1.22 | 0.26 |
| hvdc2 | 190 | 1.35 | 0.99 |
| InnerLoop1 | 197 | 0.75 | 0.44 |
| Jacobian_unbalancedLdf | 203 | 2.41 | 0.80 |
| mac_econ_fwd500 | 206 | 1.27 | 0.07 |
| rajat21 | 411 | 1.89 | 0.76 |
| Newton_iteration1 | 427 | 2.38 | 0.14 |
| esc64a | 504 | 2.40 | 0.00 |
| mc2depi | 525 | 2.10 | 0.00 |
| pre2 | 659 | 5.96 | 0.36 |

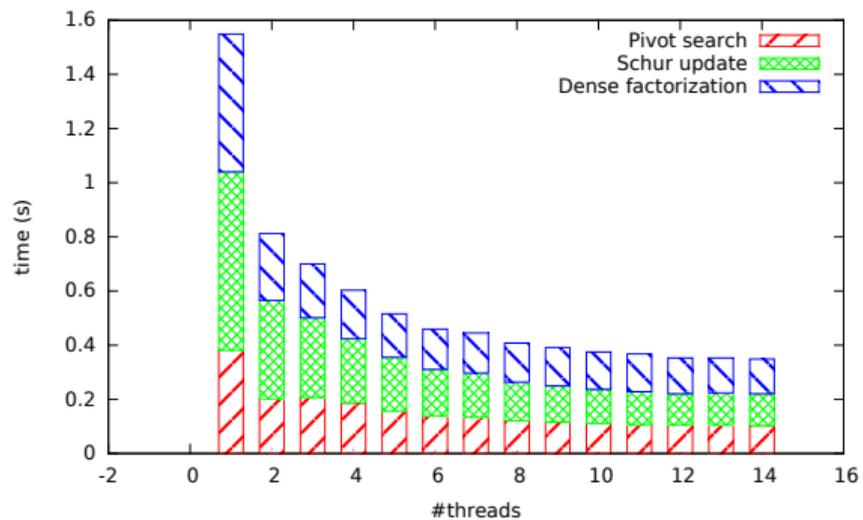
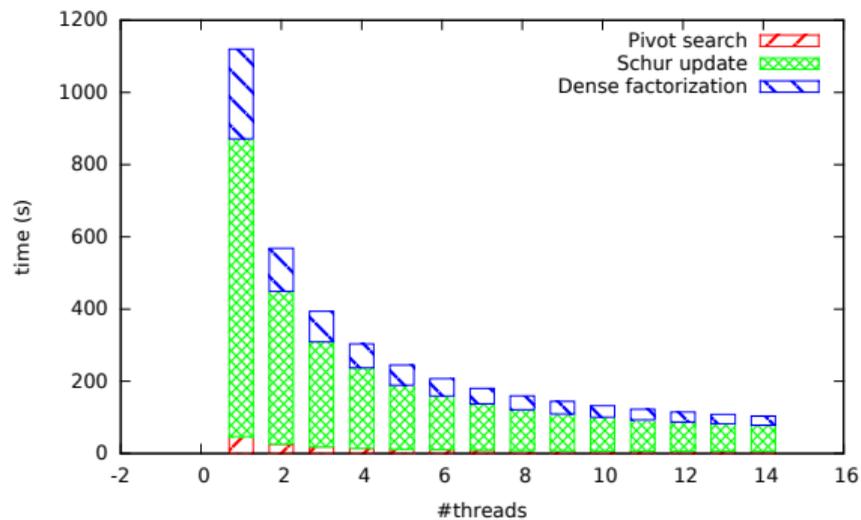
Effect of the threshold value on the backward error



Effect of the Markowitz threshold on the fill-in factor ($nz(LU)/nz(A)$)



Scalability on the mc2depi and twotone matrices



Comparison of ParSHUM, MUMPS and UMFPACK on single NUMA node (14 cores)

| Matrix | ParSHUM | | MUMPS | | UMFPACK | |
|------------------------|-------------|---------|-------------|---------|-------------|---------|
| | time | fill-in | time | fill-in | time | fill-in |
| nug30 | 0.71 | 142. | 6.19 | 730 | 48.15 | 463. |
| esc32a | 0.46 | 70.9 | 10.59 | 758 | 71.0 | 341. |
| lung2 | 0.04 | 1.48 | — | — | 0.10 | 1.44 |
| twotone | 0.35 | 7.37 | 2.22 | 25.4 | 0.49 | 5.45 |
| hvdc2 | 0.38 | 6.91 | 1.13 | 2.14 | 0.38 | 2.06 |
| InnerLoop1 | 0.13 | 2.71 | 1.64 | 3.90 | 0.30 | 2.48 |
| Jacobian_unbalancedLdf | 0.99 | 10.9 | 1.52 | 3.45 | 0.74 | 3.88 |
| mac_econ_fwd500 | 33.5 | 450. | 10.5 | 56.4 | 4.62 | 57.5 |
| rajat21 | 0.11 | 1.64 | — | — | 44.5 | 1.89 |
| Newton_Iteration1 | 0.46 | 3.46 | 4.26 | 6.02 | 1.00 | 2.55 |
| esc64a | 12.8 | 155 | — | 2151 | — | — |
| mc2depi | 104. | 302. | 4.60 | 25 | 4.36 | 38.6 |
| pre2 | 13.0 | 57.0 | 9.73 | 18.4 | 25.8 | 32.3 |

Table: The execution time and the fill-in factor for ParSHUM, MUMPS and UMFPACK.

Towards distributed memory: Singly Bordered Block decomposition using Zoltan

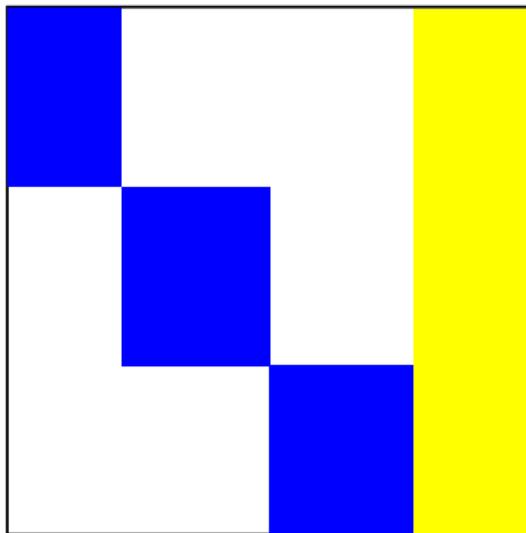
Global view



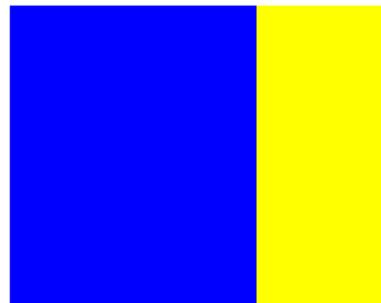
Local view

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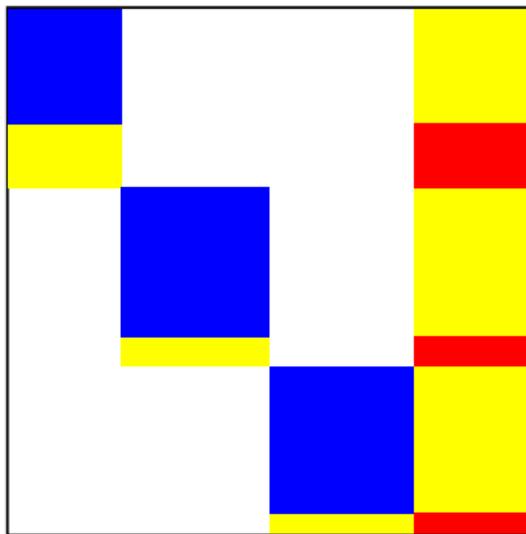


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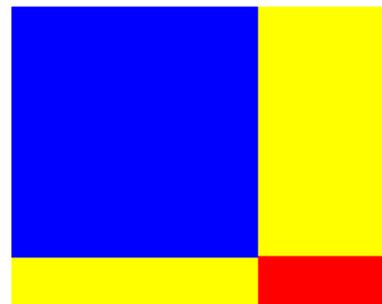


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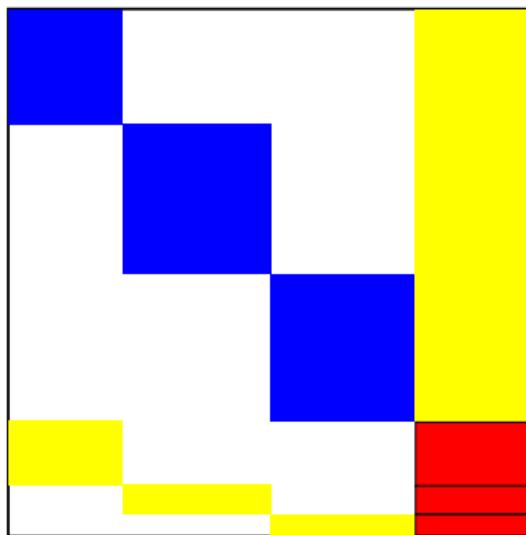


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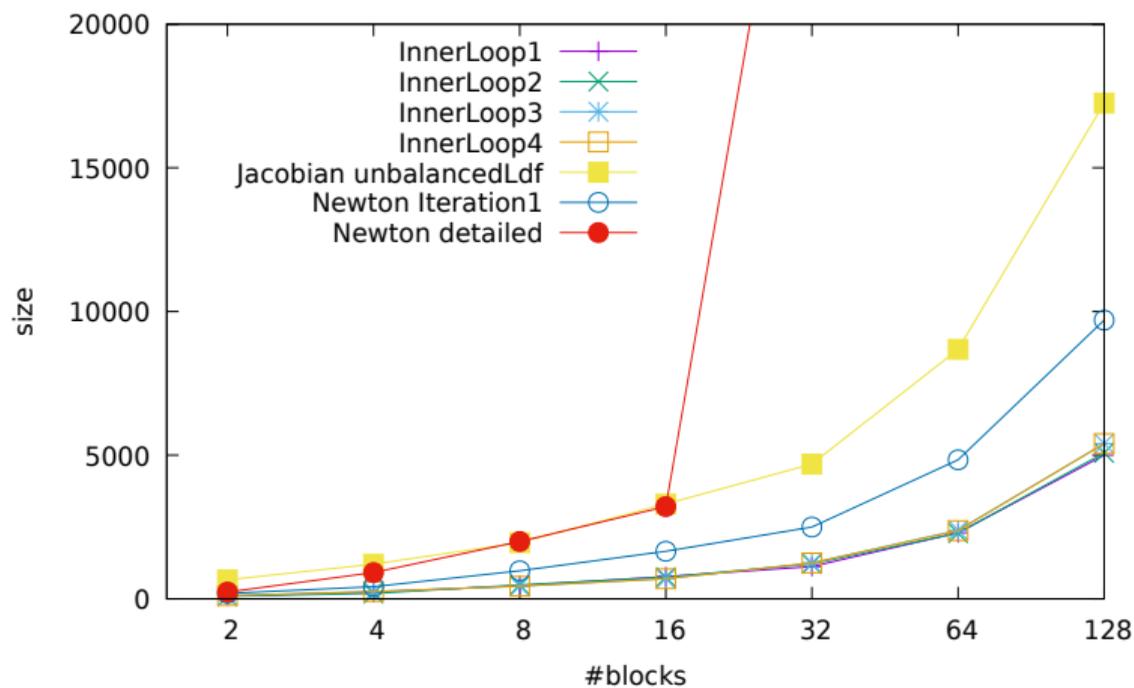
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| Matrix | Order $\times 10^3$ | Entries $\times 10^6$ | <i>si</i> |
|------------------------|------------------------|--------------------------|-----------|
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| Jacobian_unbalancedLdf | 203 | 2.41 | 0.80 |
| Newton_iteration1 | 427 | 2.38 | 0.14 |
| Newton_detailed | 7355 | 24 | 0.29 |

Bordered Block size



Scalability

| Matrix | #MPI processes | | | |
|------------------------|----------------|------|------|------|
| | 1 | 2 | 4 | 8 |
| InnerLoop1 | 0.15 | 0.08 | 0.06 | 0.05 |
| InnerLoop2 | 0.15 | 0.08 | 0.06 | 0.05 |
| InnerLoop3 | 0.15 | 0.09 | 0.06 | 0.05 |
| InnerLoop4 | 0.15 | 0.09 | 0.06 | 0.05 |
| Jacobian_unbalancedLdf | 1.11 | 0.52 | 0.35 | 0.18 |
| Newton_iteration1 | 0.49 | 0.30 | 0.14 | 0.09 |
| Newton_detailed | 7.39 | 3.01 | 1.67 | 1.63 |

The execution time in seconds for ParSHUM on the test matrices partitioned in SBBD form. One numa node (14 cores) is used per process.

Comparison with MUMPS and SuperLU

| Matrix | ParSHUM | | MUMPS | | SuperLU | |
|------------------------|-------------|---------|-------|---------|---------|---------|
| | time | fill-in | time | fill-in | time | fill-in |
| InnerLoop1 | 0.05 | 3.03 | 0.23 | 3.90 | 1.11 | 6.02 |
| InnerLoop2 | 0.05 | 2.82 | 0.25 | 3.72 | 1.11 | 5.57 |
| InnerLoop3 | 0.05 | 2.78 | 0.17 | 3.72 | 1.10 | 5.60 |
| InnerLoop4 | 0.05 | 2.41 | 0.15 | 3.73 | 1.15 | 5.56 |
| Jacobian_unbalancedLdf | 0.18 | 6.75 | 0.28 | 3.45 | 1.05 | 3.59 |
| Newton_iteration1 | 0.09 | 3.57 | 0.42 | 6.02 | 2.59 | 5.60 |
| Newton_detailed | 1.63 | 5.32 | 7.88 | 6.09 | 52.4 | 5.39 |

Conclusions and future work

Conclusions:

- ▶ We have developed a generalization of Luby's algorithm for directed graphs.
- ▶ We have used this to develop a multi-threaded threshold Markowitz code (ParSHUM library).
- ▶ In general it outperforms established codes.

Future work:

- ▶ Ongoing work on SBBD to exploit distributed systems.
- ▶ Develop a GPU version of the solver.

THANK YOU FOR YOUR ATTENTION