



Block Preconditioners for Incompressible Magnetohydrodynamics

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Joint work with Chen Greif
The University of British Columbia

Outline

Incompressible Magnetohydrodynamics Model Problem

Block Preconditioner

Approximate Inverse

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MHD Model

- ▶ MHD models electrically conductive fluids (such as liquid metals, plasma, salt water, etc) in an electromagnetic field
- ▶ Applications: electromagnetic pumping, aluminum electrolysis, the Earth's molten core and solar flares
- ▶ MHD couples electromagnetism (governed by Maxwell's equations) and fluid dynamics (governed by the Navier-Stokes equations)
 - ▶ Motion of the conductive fluid induces and modifies the existing electromagnetic field
 - ▶ Electromagnetic field generates a force on the fluid

Continuous MHD Model

Elliptic PDE in steady state

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \kappa (\nabla \times \mathbf{b}) \times \mathbf{b} = \mathbf{f} \quad \text{in } \Omega,$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega,$$

$$\kappa \nu_m \nabla \times (\nabla \times \mathbf{b}) + \nabla r - \kappa \nabla \times (\mathbf{u} \times \mathbf{b}) = \mathbf{g} \quad \text{in } \Omega,$$

$$\nabla \cdot \mathbf{b} = 0 \quad \text{in } \Omega,$$

with appropriate boundary conditions.

- ▶ $(\nabla \times \mathbf{b}) \times \mathbf{b}$: Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields
- ▶ $\nabla \times (\mathbf{u} \times \mathbf{b})$: electromotive force modifying the magnetic field

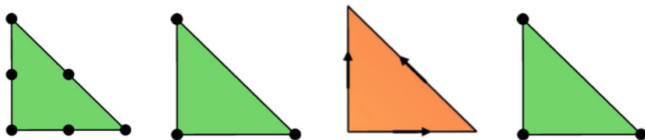
Discretization: Finite Element Method

Weak formulation: Integrate $(\mathbf{u}, p, \mathbf{b}, r)$ against a set of test functions

$$\begin{aligned} \int_{\Omega} \nu \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \kappa (\mathbf{v} \times \mathbf{b}) \cdot \nabla \times \mathbf{b} - \int_{\Omega} \nabla \cdot \mathbf{v} \cdot p &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \\ - \int_{\Omega} \nabla \cdot \mathbf{u} \cdot q &= 0, \\ \int_{\Omega} \kappa \nu_m \nabla \times \mathbf{b} \cdot \nabla \times \mathbf{c} - \int_{\Omega} \kappa (\mathbf{u} \times \mathbf{b}) \cdot \nabla \times \mathbf{b} + \int_{\Omega} \mathbf{c} \cdot \nabla r &= \int_{\Omega} \mathbf{g} \cdot \mathbf{c}, \\ \int_{\Omega} \mathbf{b} \cdot \nabla s &= 0. \end{aligned}$$

Discretization: Finite Element Method

Mixed finite elements: $H^1(\Omega) \times L^2(\Omega) \times H(\text{curl}, \Omega) \times H^1(\Omega)$



Discretized and Linearized MHD Model:

$$\left(\begin{array}{cc|cc} F & B^T & C^T & 0 \\ B & 0 & 0 & 0 \\ \hline -C & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array} \right) \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix},$$

C : coupling terms; F : convection–diffusion term; B : fluid divergence operator; M : curl-curl operator; D : magnetic divergence operator

MHD Preconditioners

- ▶ Phillips, Elman, Cyr, Shadid, and Pawlowski (2014, 2016) derived block triangular preconditioners for a block 3-by-3 and 4-by-4 formulation of the MHD model
- ▶ Adler, Benson, Cyr, MacLachlan, and Tuminaro (2016) developed an “all-at-once” type multigrid solver based on Vanka smoothers for the 4-by-4 formulation
- ▶ Wathen, G., and Schötzau (2017): Schur complement-based preconditioner
- ▶ Wathen and G. (2019): approximate inverse-based preconditioner

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Ideal preconditioning

Non-singular (1, 1) block (as in Navier-Stokes)

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & 0 \\ 0 & BF^{-1}B^T \end{pmatrix}$$

Murphy, Golub & Wathen (2000) showed that $\mathcal{P}^{-1}\mathcal{K}$ has three eigenvalues (1 and $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$). Can insert B^T in (1,2) block for nonsymmetric problem.

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$$\mathcal{K} = \begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} M + D^T W^{-1} D & 0 \\ 0 & W \end{pmatrix}, \text{ where } W \text{ is SPD}$$

M singular with nullity m (as in time-harmonic Maxwell)

G. & Schötzau (2006) showed that $\mathcal{P}^{-1}\mathcal{K}$ has exactly two eigenvalues: ± 1

Block preconditioning

Combine established preconditioners for the sub-problems

$$\mathcal{M}_I^{\text{MHD}} = \left(\begin{array}{cc|cc} F & B^T & C^T & 0 \\ 0 & -BF^{-1}B^T & 0 & 0 \\ \hline -C & 0 & M + D^T L^{-1} D & 0 \\ 0 & 0 & 0 & L \end{array} \right)$$

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Block preconditioning

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Combining fluid and magnet field using Schur complement technique

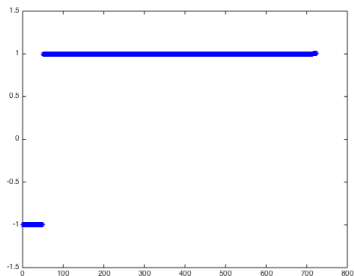
$$\mathcal{M}_S^{\text{MHD}} = \left(\begin{array}{cccc} F + M_C & C^T & B^T & 0 \\ 0 & M + D^T L^{-1} D & 0 & 0 \\ 0 & 0 & -BF^{-1}B^T & 0 \\ 0 & 0 & 0 & L \end{array} \right)$$

where $M_C = C^T(M + D^T L^{-1} D)^{-1} C$

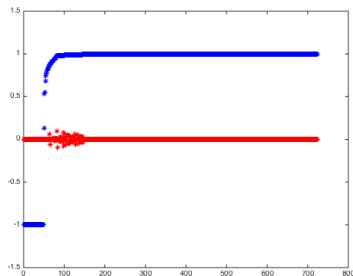
Spectral Structure: Clustering Effect

Red: imaginary part of eigenvalues

Blue: real part of eigenvalues



Eigenvalues of the preconditioned matrix $(\mathcal{M}_I^{\text{MHD}})^{-1}\mathcal{K}$



Eigenvalues of the preconditioned matrix $(\mathcal{M}_S^{\text{MHD}})^{-1}\mathcal{K}$

Numerical Software

- ▶ *FEniCS*: finite element discretization (Sweden/USA/UK)
 - ▶ *mshr*: mesh generator (utilizing *Tetgen* and *CGAL*)
- ▶ *PETSc*: linear algebra backend (Argonne National Lab)
 - ▶ *Hypre*: AMG solver (Lawrence Livermore National Lab)
 - ▶ *MUMPS*: sparse direct solver (France)

Results

ℓ	DoF	time _{solve}	time _{NL}	it _{NL}	it _{av} ^I	it _{av} ^D
1	527	0.03	0.9	4	18.8	18.0
2	3,041	0.22	3.5	3	26.7	22.3
3	20,381	1.77	26.6	3	37.0	24.7
4	148,661	22.11	237.0	3	40.7	26.0
5	1,134,437	206.43	2032.7	3	44.3	-
6	8,861,381	2274.28	19662.0	3	50.0	-

Table: 3D smooth: Number of nonlinear iterations and number of iterations to solve the MHD system with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

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Approximate Inverse

Approximate Inverse

- ▶ Estrin & G. (2015): an inverse formula for saddle-point systems with a maximally rank-deficient leading block

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

with $\text{rank}(A) = n - m$, $\text{rank}(B) = m$, and $\ker(A) \cap \ker(B) = \{0\}$.

- ▶ The mixed Maxwell formulation used falls into this class of saddle-point systems
- ▶ If the leading block has a maximal nullity then the inverse has a zero (2,2) block and the other blocks can be represented by the null-space of the leading block

Discretized and Linearized Equations

Back to the equations we solve in the nonlinear iteration (with a slight change of notation):

$$\left(\begin{array}{cc|cc} F(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ \hline -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array} \right) \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix} \left. \begin{array}{l} \} n_u \text{ rows} \\ \} m_u \text{ rows} \\ \} n_b \text{ rows} \\ \} m_b \text{ rows} \end{array} \right\}$$

Let

$$\mathcal{K}x = \begin{pmatrix} \mathcal{K}_{NS} & \mathcal{K}_C^T \\ -\mathcal{K}_C & \mathcal{K}_M \end{pmatrix} \begin{pmatrix} x_v \\ x_b \end{pmatrix} = \begin{pmatrix} f_u \\ f_b \end{pmatrix}$$

where \mathcal{K}_{NS} , \mathcal{K}_C and \mathcal{K}_M are the Navier-Stokes, coupling and Maxwell block matrices

General Inverse of Saddle Point Matrix

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

$$A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{m \times n}$$

If A is non-singular then from Benzi, Golub & Liesen (2005)

$$\mathcal{A}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B^T S^{-1}BA^{-1} & -A^{-1}B^T S^{-1} \\ -S^{-1}BA^{-1} & S^{-1} \end{pmatrix}$$

Will be useful for the Navier-Stokes block

Maximal Nullity of Leading Block

Estrin & G. (2015): If A has nullity m then the (2,2) block of the inverse is zero. The inverse formula is

$$\mathcal{A}^{-1} = \begin{pmatrix} A_W^{-1}(I - D^T W^{-1} G^T) & G W^{-1} \\ W^{-1} G^T & 0 \end{pmatrix},$$

where W is a (free) symmetric positive definite matrix,

$$A_W = A + B^T W^{-1} B \quad \text{and} \quad G = A_W^{-1} B^T.$$

This comes handy for the block Schur complement associated with the 4×4 block MHD matrix

Block Schur Complement

Then

$$\mathcal{S} = \mathcal{K}_M + \mathcal{K}_C \mathcal{K}_{NS}^{-1} \mathcal{K}_C^T = \begin{pmatrix} M + CK_1C^T & D^T \\ D & 0 \end{pmatrix}$$

Remark

Note that the null C^T and M have the same null space (discrete gradients). Therefore,

$$\dim(\text{null}(M + CK_1C^T)) = m_b,$$

where m_b is the number of rows of the magnetic discrete divergence matrix D

Block Schur Complement

Then

$$S = K_M + K_C K_{NS}^{-1} K_C^T = \begin{pmatrix} M + CK_1 C^T & D^T \\ D & 0 \end{pmatrix}$$

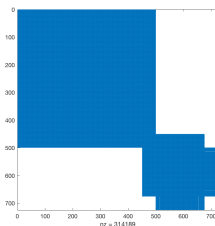
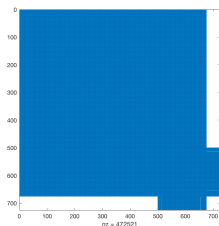
Using Estrin & G. (2015):

$$S^{-1} = \begin{pmatrix} M_F^{-1}(I - D^T W^{-1} G^T) & G W^{-1} \\ W^{-1} G^T & 0 \end{pmatrix},$$

where W is a (free) symmetric positive definite matrix,

$$M_F = M + D^T W^{-1} D + CK_1 C^T \quad \text{and} \quad G = M_F^{-1} D^T.$$

Sparse Block Approximation



Sparsify utilizing:

1. Small mesh-based block elements
2. Null-space properties
3. Approximate Schur complements

Note: Never explicitly form the dense blocks

Eigenvalue Distribution

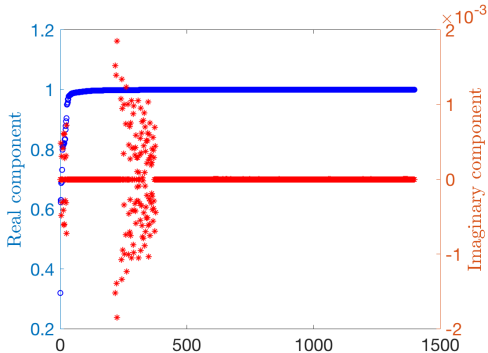


Figure: Eigenvalues of preconditioned matrix $\mathcal{P}_1^{-1} \mathcal{K}$ where red are the imaginary and blue are the real parts of the eigenvalues.

3D Cavity Driven Flow

$$\begin{aligned}\mathbf{u} &= (1, 0, 0) && \text{on } z = 1, \\ \mathbf{u} &= (0, 0, 0) && \text{on } x = \pm 1, y = \pm 1, z = -1, \\ \mathbf{n} \times \mathbf{b} &= \mathbf{n} \times \mathbf{b}_N && \text{on } \partial\Omega, \\ r &= 0 && \text{on } \partial\Omega,\end{aligned}$$

where $\mathbf{b}_N = (-1, 0, 0)$.

3D Cavity Driven Flow

ℓ	DoFs	time ^A	it_{NL}^A	it_O^A
1	14,012	7.58	4	57.0
2	28,436	22.21	4	56.2
3	64,697	65.95	4	56.0
4	245,276	271.48	4	56.0
5	937,715	1255.15	4	55.5
6	5,057,636	17656.36	4	58.5

Table: 3D Cavity Driven using both the approximate inverse and block triangular preconditioner with parameters $\kappa = 1e1$, $\nu = 1e-1$, $\nu_m = 1e-1$ and $Ha = \sqrt{1000}$

Fichera Corner

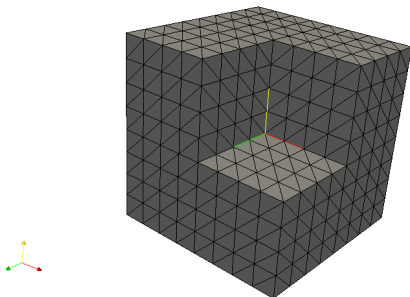


Figure: Example Fichera corner domain for mesh level $\ell = 3$

Fichera Corner

ℓ	DoFs	time ^A	it_{NL}^A	it_O^A
1	34,250	15.64	4	29.2
2	57,569	30.41	4	29.2
3	89,612	52.90	4	28.8
4	332,744	232.23	4	27.8
5	999,269	1026.31	4	27.8
6	5,232,365	11593.47	5	28.6

Table: Fichera corner using the approximate inverse preconditioner $\kappa = 1e1$, $\nu = 1e-2$, $\nu_m = 1e-2$ and $Ha = \sqrt{1e5}$.

References

- ▶ *Preconditioners for Mixed Finite Element Discretizations of Incompressible MHD Equations*, Michael Wathen, Chen Greif, and Dominik Schötzau, SIAM Journal on Scientific Computing, 39(6):A2293-A3013, 2017.
- ▶ *A Scalable Approximate Inverse Block Preconditioner for an Incompressible Magnetohydrodynamics Model Problem*, Michael Wathen and Chen Greif, in review

Thank you!