Preconditioning for nonsymmetric systems

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Krylov subspace methods

For self-adjoint problems/symmetric matrices, iterative methods of choice exist: conjugate gradients for SPD, MINRES otherwise

but many possible methods for non-self-adjoint problems/nonsymmetric matrices: GMRES , BICGSTAB , QMR , IDR , . . .

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For almost all need *preconditioning*

Preconditioner $\mathbf{P}$ such that

$$\text{“}\mathbf{P}^{-1}\mathbf{B}\mathbf{x} = \mathbf{P}^{-1}\mathbf{b}\text{”}$$

has much faster convergence with the appropriate iterative method than $\mathbf{B}\mathbf{x} = \mathbf{b}$.

Krylov subspace methods

Arises because of convergence guarantees:

- for symmetric matrices: descriptive convergence bounds based on eigenvalues $\Rightarrow$ a priori estimates of iterations for acceptable convergence; good preconditioning ensures fast convergence.
- for nonsymmetric matrices: by contrast, to date there are no generally applicable *and descriptive* convergence bounds even for GMRES ; for any of the other nonsymmetric methods without a minimisation property, convergence theory is extremely limited $\Rightarrow$ no good a priori way to identify what are the desired qualities of a preconditioner

A major theoretical difficulty, but heuristic ideas abound!

Real Toeplitz matrices

can use *Pestana & W, 2015*: If **B** is a real Toeplitz matrix then

$$\underbrace{\begin{bmatrix} a_0 & a_{-1} & \cdot & \cdot & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdot & \cdot \\ \cdot & a_1 & a_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{-1} \\ a_{n-1} & \cdot & \cdot & a_1 & a_0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} 0 & 0 & \cdot & 0 & 1 \\ 0 & \cdot & 0 & 1 & 0 \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & 0 & 0 \end{bmatrix}}_{\mathbf{Y}}$$

is the real *symmetric* (Hankel) matrix

$$\begin{bmatrix} a_{1-n} & \cdot & \cdot & a_{-1} & a_0 \\ \cdot & \cdot & a_{-1} & a_0 & a_1 \\ \cdot & \cdot & a_0 & a_1 & \cdot \\ a_{-1} & \cdot & \cdot & \cdot & \cdot \\ a_0 & a_1 & \cdot & \cdot & a_{n-1} \end{bmatrix}$$

so can use MINRES with positive definite preconditioner

Toeplitz $\rightarrow$ Circulant approximation, C, and diagonalization $C = F^* \Lambda F$ by the FFT ($O(n \log n)$) is a well known approach.
To ensure positive definiteness, take the absolute value circulant $|C| = F^* |\Lambda| F$

Theorem (*Pestana & W, 2015*)

$$|\mathbf{C}|^{-1}\mathbf{B}\mathbf{Y} = \mathbf{J} + \mathbf{R} + \mathbf{E}$$

where $\mathbf{J}$ is real symmetric and orthogonal with eigenvalues ± 1, $\mathbf{R}$ is of small rank and $\mathbf{E}$ is of small norm

$\Rightarrow$ guaranteed fast convergence because MINRES convergence only depends on eigenvalues which are clustered around ± 1 except for few outliers!

'Krylov friendly' $\rightarrow$ try GMRES simply with C; generally converges in half the number of iterations

Natural to investigate how sensitive to exact Toeplitz structure.

Our simple test problem is the IVP:

$$y'(t) = ay(t) + g(t), \qquad y(0) = y_0, \;\; t \in [0, T]$$

approximated by the θ-method:

$$(-1 - a(1-\theta)h)y_n + (1 - a\theta h)y_{n+1} = hg(t_n)$$

and the BDF2 nethod

$$y_{n+2} - (4/3)y_{n+1} + (1/3)y_n = (2/3)hg(t_{n+2})$$

where for a regular mesh we have

$$t_n = nh, n = 0, 1, \ldots, N \quad \text{and} \quad Nh = T.$$

Both are written in monolithic (all-at-once) form yielding

$$\boldsymbol{Bx = b}$$

where for example for the $\boldsymbol{\theta}$-method with a regular grid we have the exact Toeplitz matrix

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b} & & & & \\ \boldsymbol{c} & \boldsymbol{b} & & & \\ & \boldsymbol{c} & \boldsymbol{b} & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{c} & \boldsymbol{b} \end{bmatrix},$$

$\boldsymbol{b = 1 - a\theta h}$, $\boldsymbol{c = -1 - a(1-\theta)h}$ and the vector $\boldsymbol{x}$ contains the solution at all time steps.

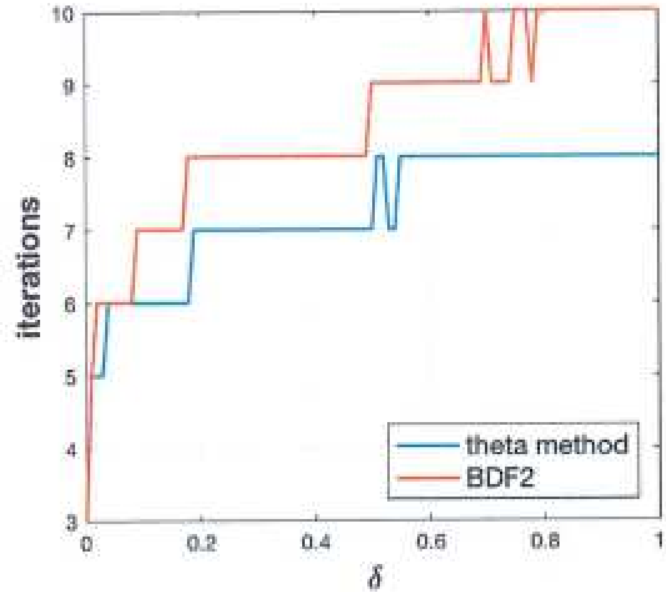
For an irregular grid we set

$$t_n = n\,h + \delta\, Rand\, h, \qquad n = 1, 2, \ldots, N-1$$

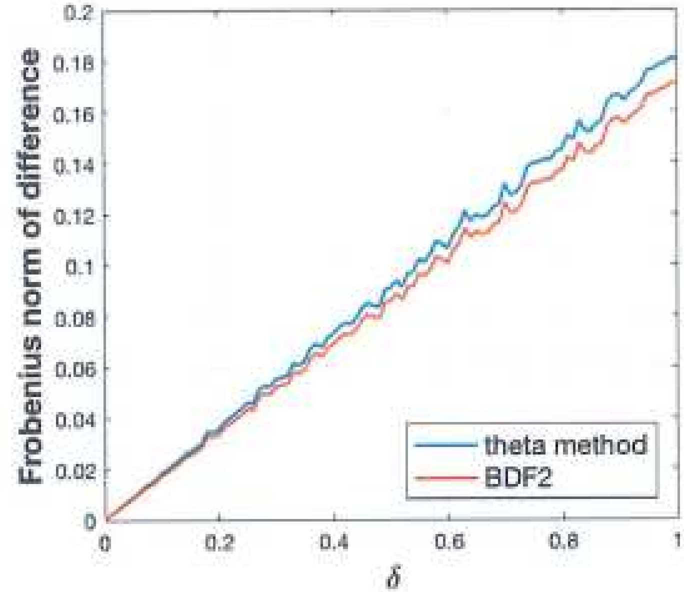
where $Rand$ is, for each n, selected from a uniform random distribution on $[-1/2, 1/2]$.

$\delta = 0$ gives a regular grid and therefore a Toeplitz matrix, whereas increasingly irregular grids result for larger values of δ. For increasing δ the coefficient matrix is further from Toeplitz.

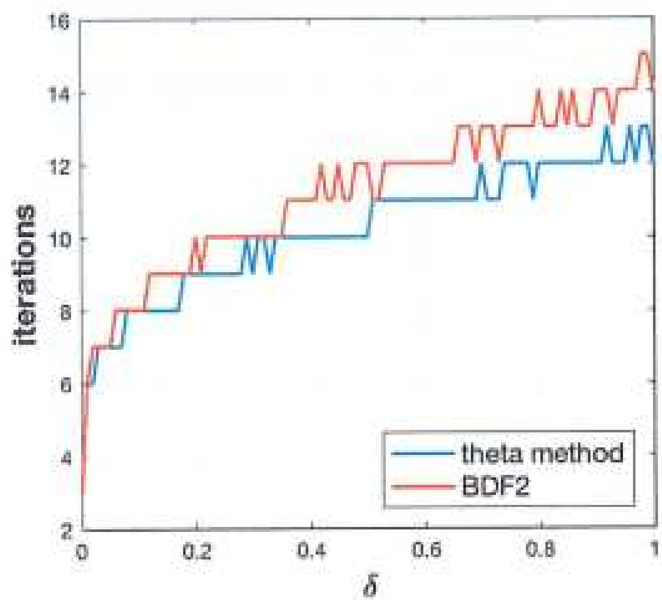
We precondition with the Strang circulant for the matrix obtained by averaging along each diagonal and solve for $T = 10$ with $h = 0.01$ and $h = 0.1$.



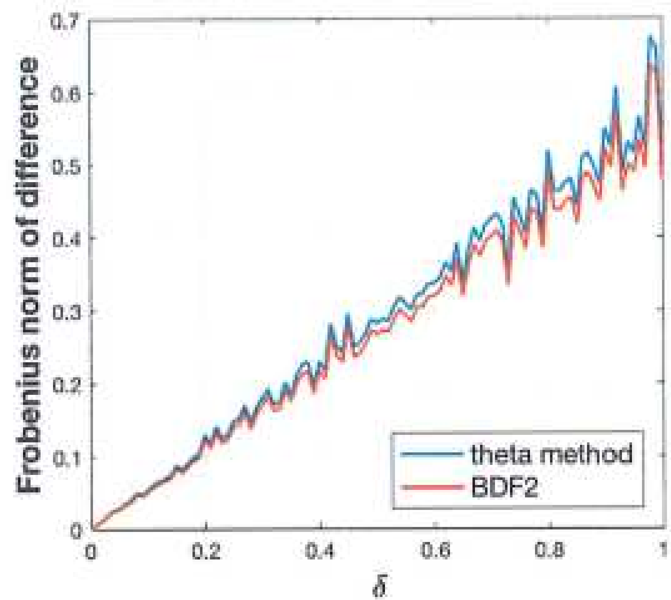
(a) Number of iterations of GMRES ($h = 0.01$).



(b) Frobenius norm of D ($h = 0.01$).



(c) Number of iterations of GMRES ($h = 0.1$).



(d) Frobenius norm of D ($h = 0.1$).

For a genuinely irregular grid—100 grid points generated randomly in $[0, T]$—100 iterations are required as might be expected for a Krylov method.

In a further experiment we randomly generate matrices of dimension 100 with differing distance from a circulant:

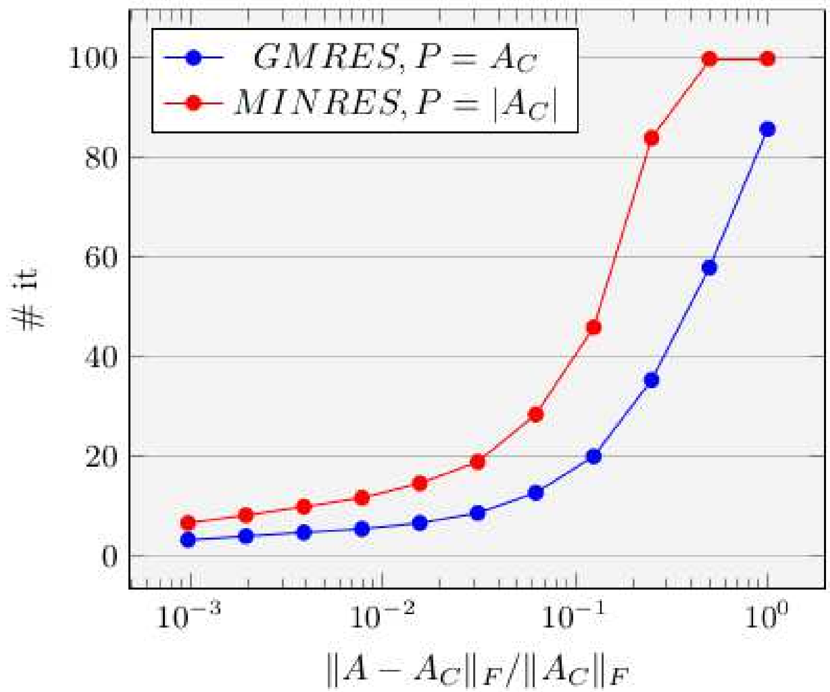


Figure 4: Average number of GMRES (blue) and MINRES (red) iterations to convergence applied to 20 dense symmetric randomly generated (uniform distributions) matrices A of size 100. As a preconditioner, the closest Circulant A_C is used for GMRES, while its absolute value $|A_C|$ for MINRES. Different values of the relative norm of the non-Circulant component of A, $\|A - A_C\|_F / \|A_C\|_F$ are considered. Convergence reached with residual norm $\leq 10^{-6}$.

Summary

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Reference:

Pestana, J. & Wathen, A.J., 2015,
'A preconditioned MINRES method for nonsymmetric Toeplitz matrices',

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