Block Preconditioners for Incompressible Magnetohydrodynamics

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Joint work with Chen Greif
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Outline

Incompressible Magnetohydrodynamics Model Problem

Block Preconditioner

Approximate Inverse

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MHD Model

- MHD models electrically conductive fluids (such as liquid metals, plasma, salt water, etc) in an electromagnetic field
- Applications: electromagnetic pumping, aluminum electrolysis, the Earth's molten core and solar flares
- MHD couples electromagnetism (governed by Maxwell's equations) and fluid dynamics (governed by the Navier-Stokes equations)
  - Motion of the conductive fluid induces and modifies the existing electromagnetic field
  - Electromagnetic field generates a force on the fluid

Continuous MHD Model

Elliptic PDE in steady state

$$
\begin{aligned}
-\nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \kappa\,(\nabla\times\mathbf{b})\times\mathbf{b} &= \mathbf{f} && \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega,\\
\kappa\nu_m\,\nabla\times(\nabla\times\mathbf{b}) + \nabla r - \kappa\,\nabla\times(\mathbf{u}\times\mathbf{b}) &= \mathbf{g} && \text{in } \Omega,\\
\nabla\cdot\mathbf{b} &= 0 && \text{in } \Omega,
\end{aligned}
$$

with appropriate boundary conditions.

- $(\nabla \times b) \times b$: Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields
- $\nabla \times (u \times b)$: electromotive force modifying the magnetic field

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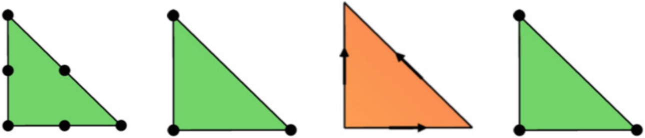
Discretization: Finite Element Method

Weak formulation: Integrate $(\mathbf{u}, p, \mathbf{b}, r)$ against a set of test functions

$$\begin{aligned}
\int_\Omega \nu \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} + \int_\Omega \kappa\, (\mathbf{v} \times \mathbf{b}) \cdot \nabla \times \mathbf{b} - \int_\Omega \nabla \cdot \mathbf{v} \cdot\ p &= \int_\Omega \mathbf{f} \cdot \mathbf{v}, \\
-\int_\Omega \nabla \cdot \mathbf{u} \cdot\ q &= 0, \\
\int_\Omega \kappa \nu_m \nabla \times \mathbf{b} \cdot \nabla \times \mathbf{c} - \int_\Omega \kappa\, (\mathbf{u} \times \mathbf{b}) \cdot \nabla \times \mathbf{b} + \int_\Omega \mathbf{c} \cdot \nabla r &= \int_\Omega \mathbf{g} \cdot \mathbf{c}, \\
\int_\Omega \mathbf{b} \cdot \nabla s &= 0.
\end{aligned}$$

Discretization: Finite Element Method

Mixed finite elements: $H^1(\Omega) \times L^2(\Omega) \times H(\mathrm{curl}, \Omega) \times H^1(\Omega)$



Discretized and Linearized MHD Model:

$$\left(\begin{array}{cc|cc} F & B^T & C^T & 0 \\ B & 0 & 0 & 0 \\ \hline -C & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right) \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix},$$

C: coupling terms; F: convection–diffusion term; B: fluid divergence operator; M: curl-curl operator; D: magnetic divergence operator

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MHD Preconditioners

- Phillips, Elman, Cyr, Shadid, and Pawlowski (2014, 2016) derived block triangular preconditioners for a block 3-by-3 and 4-by-4 formulation of the MHD model
- Adler, Benson, Cyr, MacLachlan, and Tuminaro (2016) developed an "all-at-once" type multigrid solver based on Vanka smoothers for the 4-by-4 formulation
- Wathen, G., and Schötzau (2017): Schur complement-based preconditioner
- Wathen and G. (2019): approximate inverse-based preconditioner

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Ideal preconditioning

Non-singular $(1,1)$ block (as in Navier-Stokes)

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & 0 \\ 0 & BF^{-1}B^T \end{pmatrix}$$

Murphy, Golub & Wathen (2000) showed that $\mathcal{P}^{-1}\mathcal{K}$ has three eigenvalues (1 and $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$). Can insert B^T in (1,2) block for nonsymmetric problem.

Ideal preconditioning

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$$\mathcal{K} = \begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} M + D^T W^{-1} D & 0 \\ 0 & W \end{pmatrix}, \text{ where } W \text{ is SPD}$$

M singular with nullity m (as in time-harmonic Maxwell)

G. & Schötzau (2006) showed that $\mathcal{P}^{-1}\mathcal{K}$ has exactly two eigenvalues: ± 1

Block preconditioning

Combine established preconditioners for the sub-problems

$$\mathcal{M}_{\mathrm{I}}^{\mathrm{MHD}} = \left(\begin{array}{cc|cc} F & B^T & C^T & 0 \\ 0 & -BF^{-1}B^T & 0 & 0 \\ \hline -C & 0 & M + D^T L^{-1} D & 0 \\ 0 & 0 & 0 & L \end{array}\right)$$

Block preconditioning

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$$\mathcal{M}_{\mathrm{I}}^{\mathrm{MHD}} = \left(\begin{array}{cc|cc} F & C^T & B^T & 0 \\ -C & M + D^T L^{-1} D & 0 & 0 \\ \hline 0 & 0 & -BF^{-1}B^T & 0 \\ 0 & 0 & 0 & L \end{array}\right).$$

Block preconditioning

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Combining fluid and magnet field using Schur complement technique

$$\mathcal{M}_{\mathrm{S}}^{\mathrm{MHD}} = \begin{pmatrix} F + M_C & C^T & B^T & 0 \\ 0 & M + D^T L^{-1} D & 0 & 0 \\ 0 & 0 & -BF^{-1}B^T & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

where $M_C = C^T(M + D^T L^{-1} D)^{-1} C$

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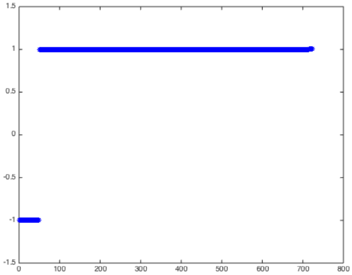
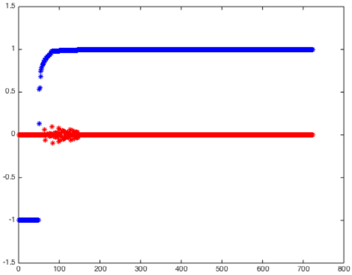
Spectral Structure: Clustering Effect

Red: imaginary part of eigenvalues

Blue: real part of eigenvalues





Eigenvalues of the preconditioned matrix $(\mathcal{M}_{\mathrm{I}}^{\mathrm{MHD}})^{-1}\mathcal{K}$

Eigenvalues of the preconditioned matrix $(\mathcal{M}_{\mathrm{S}}^{\mathrm{MHD}})^{-1}\mathcal{K}$

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Numerical Software

- *FEniCS*: finite element discretization (Sweden/USA/UK)
  - *mshr*: mesh generator (utilizing *Tetgen* and *CGAL*)
- *PETSc*: linear algebra backend (Argonne National Lab)
  - *Hypre*: AMG solver (Lawrence Livermore National Lab)
  - *MUMPS*: sparse direct solver (France)

Results

ℓ	DoF	$\text{time}_{\text{solve}}$	time_{NL}	it_{NL}	$\text{it}_{\text{av}}^{\text{I}}$	$\text{it}_{\text{av}}^{D}$
1	527	0.03	0.9	4	18.8	18.0
2	3,041	0.22	3.5	3	26.7	22.3
3	20,381	1.77	26.6	3	37.0	24.7
4	148,661	22.11	237.0	3	40.7	26.0
5	1,134,437	206.43	2032.7	3	44.3	-
6	8,861,381	2274.28	19662.0	3	50.0	-

Table: 3D smooth: Number of nonlinear iterations and number of iterations to solve the MHD system with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

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Approximate Inverse

Approximate Inverse

- Estrin & G. (2015): an inverse formula for saddle-point systems with a maximally rank-deficient leading block

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

  with $\mathrm{rank}(A) = n - m$, $\mathrm{rank}(B) = m$, and $\ker(A) \cap \ker(B) = \{0\}$.
- The mixed Maxwell formulation used falls into this class of saddle-point systems
- If the leading block has a maximal nullity then the inverse has a zero (2,2) block and the other blocks can be represented by the null-space of the leading block

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Discretized and Linearized Equations

Back to the equations we solve in the nonlinear iteration (with a slight change of notation):

$$\left(\begin{array}{cc|cc} F(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ \hline -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right) \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix} \begin{array}{l} \} \; n_u \text{ rows} \\ \} \; m_u \text{ rows} \\ \} \; n_b \text{ rows} \\ \} \; m_b \text{ rows} \end{array}$$

Let

$$\mathcal{K}x = \begin{pmatrix} \mathcal{K}_{\mathrm{NS}} & \mathcal{K}_{\mathrm{C}}^T \\ -\mathcal{K}_{\mathrm{C}} & \mathcal{K}_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} x_v \\ x_b \end{pmatrix} = \begin{pmatrix} f_u \\ f_b \end{pmatrix}$$

where $\mathcal{K}_{\mathrm{NS}}$, $\mathcal{K}_{\mathrm{C}}$ and $\mathcal{K}_{\mathrm{M}}$ are the Navier-Stokes, coupling and Maxwell block matrices

General Inverse of Saddle Point Matrix

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

$$A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{m \times n}$$

If A is non-singular then from Benzi, Golub & Liesen (2005)

$$\mathcal{A}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B^T\mathcal{S}^{-1}BA^{-1} & -A^{-1}B^T\mathcal{S}^{-1} \\ -\mathcal{S}^{-1}BA^{-1} & \mathcal{S}^{-1} \end{pmatrix}$$

Will be useful for the Navier-Stokes block

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Maximal Nullity of Leading Block

Estrin & G. (2015): If A has nullity m then the (2,2) block of the inverse is zero. The inverse formula is

$$\mathcal{A}^{-1} = \begin{pmatrix} A_W^{-1}(I - D^T W^{-1} G^T) & GW^{-1} \\ W^{-1} G^T & 0 \end{pmatrix},$$

where W is a (free) symmetric positive definite matrix,

$$A_W = A + B^T W^{-1} B \quad \text{and} \quad G = A_W^{-1} B^T.$$

This comes handy for the block Schur complement associated with the 4×4 block MHD matrix

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Block Schur Complement

Then

$$\mathcal{S} = \mathcal{K}_{\mathrm{M}} + \mathcal{K}_{\mathrm{C}}\mathcal{K}_{\mathrm{NS}}^{-1}\mathcal{K}_{\mathrm{C}}^{T} = \begin{pmatrix} M + CK_1C^T & D^T \\ D & 0 \end{pmatrix}$$

Remark

Note that the null C^T and M have the same null space (discrete gradients). Therefore,

$$dim(null(M + CK_1C^T)) = m_b,$$

where m_b is the number of rows of the magnetic discrete divergence matrix D

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Block Schur Complement

Then

$$\mathcal{S} = \mathcal{K}_{\mathrm{M}} + \mathcal{K}_{\mathrm{C}}\mathcal{K}_{\mathrm{NS}}^{-1}\mathcal{K}_{\mathrm{C}}^{T} = \begin{pmatrix} M + CK_1C^T & D^T \\ D & 0 \end{pmatrix}$$

Using Estrin & G. (2015):

$$\mathcal{S}^{-1} = \begin{pmatrix} M_F^{-1}(I - D^T W^{-1} G^T) & GW^{-1} \\ W^{-1}G^T & 0 \end{pmatrix},$$

where W is a (free) symmetric positive definite matrix,

$$M_F = M + D^T W^{-1} D + CK_1C^T \quad \text{and} \quad G = M_F^{-1} D^T.$$

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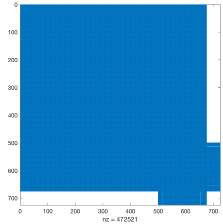
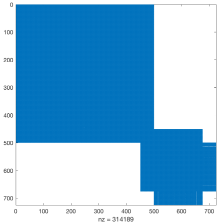
Sparse Block Approximation





Sparsify utilizing:

1. Small mesh-based block elements
2. Null-space properties
3. Approximate Schur complements

Note: Never explicitly form the dense blocks

Eigenvalue Distribution

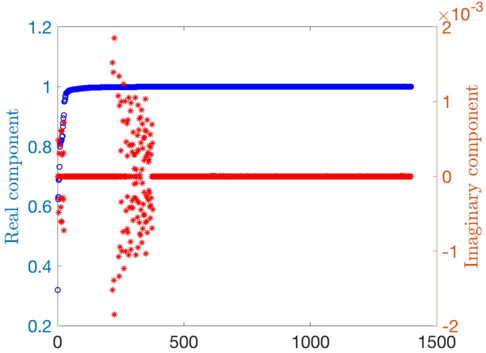


Figure: Eigenvalues of preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{K}$ where red are the imaginary and blue are the real parts of the eigenvalues.

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3D Cavity Driven Flow

$$
\begin{aligned}
&\mathbf{u} = (1,0,0) \quad \text{on} \quad z = 1, \\
&\mathbf{u} = (0,0,0) \quad \text{on} \quad x = \pm 1,\ y = \pm 1,\ z = -1, \\
&\mathbf{n} \times \mathbf{b} = \mathbf{n} \times \mathbf{b}_N \quad \text{on} \quad \partial\Omega, \\
&r = 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
$$

where $\mathbf{b}_N = (-1,0,0)$.

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3D Cavity Driven Flow

ℓ	DoFs	time$^{\mathrm{A}}$	$it_{\mathrm{NL}}^{\mathrm{A}}$	$it_{\mathrm{O}}^{\mathrm{A}}$
1	14,012	7.58	4	57.0
2	28,436	22.21	4	56.2
3	64,697	65.95	4	56.0
4	245,276	271.48	4	56.0
5	937,715	1255.15	4	55.5
6	5,057,636	17656.36	4	58.5

Table: 3D Cavity Driven using both the approximate inverse and block triangular preconditioner with parameters $\kappa = 1\mathrm{e}1$, $\nu = 1\mathrm{e}{-1}$, $\nu_m = 1\mathrm{e}{-1}$ and $\mathrm{Ha} = \sqrt{1000}$

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Fichera Corner

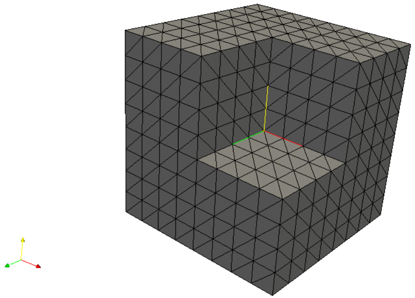

Figure: Example Fichera corner domain for mesh level $\ell = 3$

Fichera Corner

ℓ	DoFs	time^{A}	$it_{\text{NL}}^{\text{A}}$	it_{O}^{A}
1	34,250	15.64	4	29.2
2	57,569	30.41	4	29.2
3	89,612	52.90	4	28.8
4	332,744	232.23	4	27.8
5	999,269	1026.31	4	27.8
6	5,232,365	11593.47	5	28.6

Table: Fichera corner using the approximate inverse preconditioner $\kappa = 1e1$, $\nu = 1e\text{-}2$, $\nu_m = 1e\text{-}2$ and $\text{Ha} = \sqrt{1e5}$.

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References

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- *A Scalable Approximate Inverse Block Preconditioner for an Incompressible Magnetohydrodynamics Model Problem*, Michael Wathen and Chen Greif, in review

Thank you!

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