Sparse Roundoff-Error-Free LU Update

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November 24, 2020

Introduction: What is the new factorization?

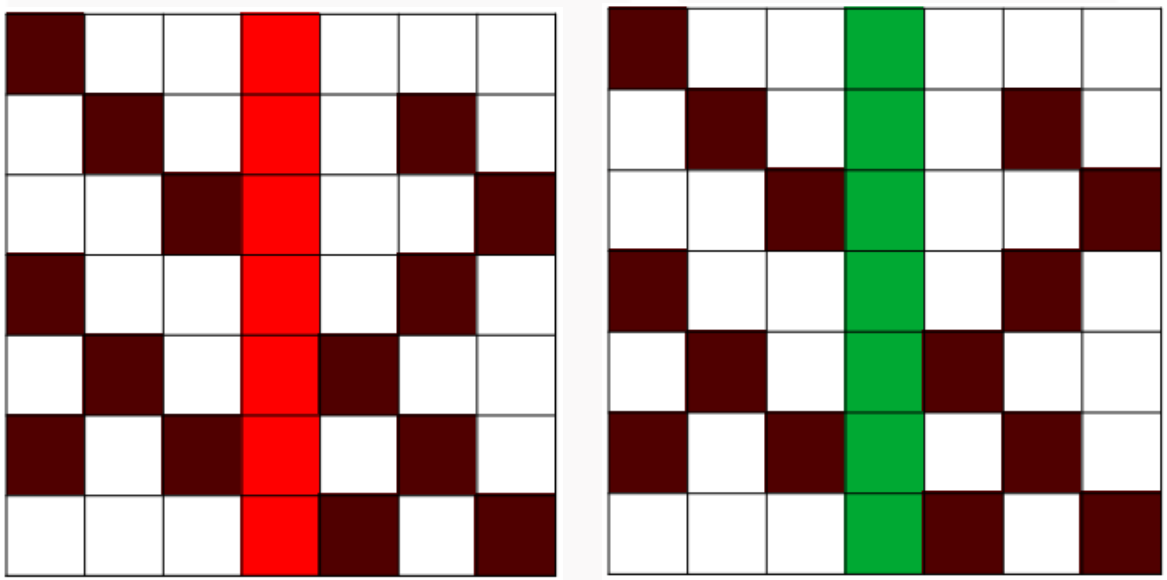


Figure: Replace a single column of A ($A(:,k)$) with a new vector v Left: original matrix A, Right: new matrix $\bar{A}$.

Gaussian Elimination v.s. IPGE

Given a submatrix $\begin{bmatrix} a_{k-1,k-1} & \cdot & \cdot \\ \cdot & a_{k,k} & a_{k,j} \\ \cdot & a_{i,k} & a_{i,j} \end{bmatrix}$, the row reduction using Gaussian elimination is done as

$$a_{i,j} = \frac{a_{i,j}a_{k,k} - a_{k,j}a_{i,k}}{a_{k,k}}$$

While using IPGE (integer-preserving Gaussian elimination) is

$$a_{i,j} = \frac{a_{i,j}a_{k,k} - a_{k,j}a_{i,k}}{a_{k-1,k-1}}$$

Properties of IPGE

1. IPGE guarantees **exact division**, and keeps entries in integer domain.
2. Let $a_{i,j}^k$ be the result of k-th IPGE iteration, it is the determinant of the submatrix induced by rows 1 to k and i and columns 1 to k and j of A.

Conventional Method for GE-based LU Update

Let k be the index of the column of A to be replaced, and v is the new column, y is the solution for $Ly = v$.

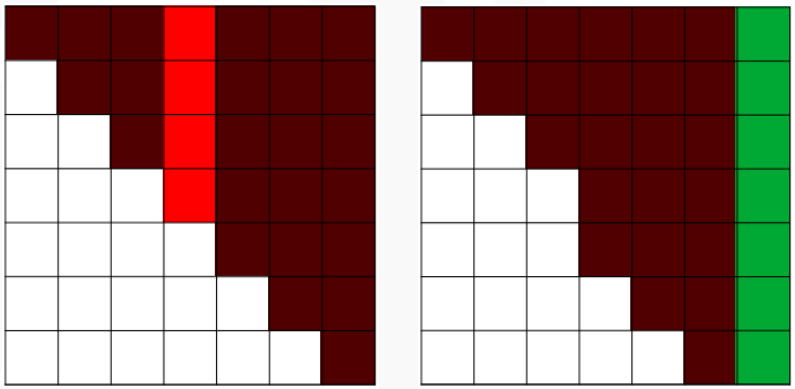


Figure: Left: original U, Right: updated U.

Conventional LU update algorithm is inadequate for the REF framework. [1]

[1] A. R. Escobedo and E. Moreno-Centeno, "Roundoff-error-free basis updates of LU factorizations for the efficient validation of optimality certificates," SIAM Journal on Matrix Analysis and Applications, vol. 38, no. 3, pp. 829–853, 2017.

Dense REF LU Update

Let k be the index of the column of A to be replaced, and v is the new column, y is the solution for $Ly = v$.

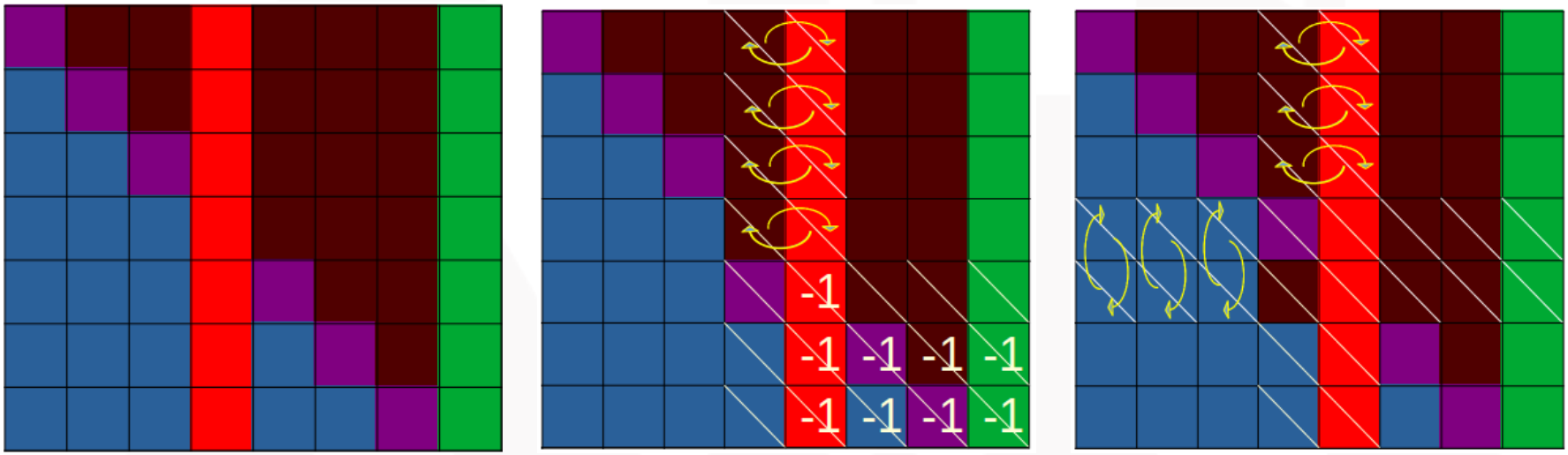


Figure: Left: original frame matrix F, Middle: updated F with APCPU [2], Right: updated F with APDPU [3].

[2] adjacent pivot column permutation update algorithm
[3] adjacent pivot diagonal permutation update algorithm

Dense REF LU Update

$$\begin{bmatrix} a^{k-2}_{k-1,k-1} & . & . \\ . & a^{k-1}_{k,k} & a^{k-1}_{k,j} \\ . & a^{k-1}_{i,k} & a^{k}_{i,j} \end{bmatrix}$$

recall the IPGE equation:

$$a_{i,j} = \frac{a_{i,j}a_{k,k} - a_{k,j}a_{i,k}}{a_{k-1,k-1}}$$

Theorem (prior dense case: backtracking)

For $k \in (0, min(i,j))$, the $(k-1)$-iteration entry $a^{k-1}_{i,j}$ can be obtained without roundoff error from its corresponding k-iterate, $a^{k}_{i,j}$, as follows:

$$a^{k-1}_{i,j} = \frac{a^{k}_{i,j}a^{k-2}_{k-1,k-1} + a^{k-1}_{k,j}a^{k-1}_{i,k}}{a^{k-1}_{k,k}}.$$

Dense REF LU Update

Let k be the index of the column of A to be replaced, and v is the new column, y is the solution for $Ly = v$.

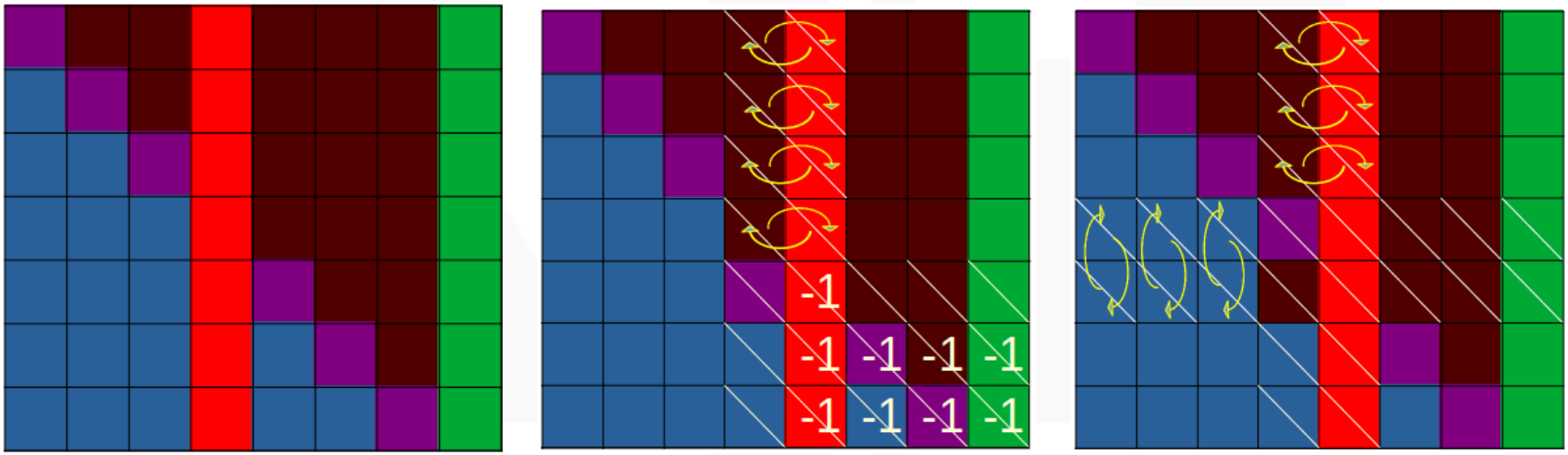


Figure: Left: original frame matrix F, Middle: updated F with APCPU, Right: updated F with APDPU.

Dense REF LU Update

Theorem (prior dense case: Row-wise Switches of Originating Pivot)

Let ρ' be a feasible pivot sequence with pivot selections identical to those of ρ in iterations 1 to $k-1$ but with the kth pivot selection from the same row but from column $k_s > k$. Let $\rho^k := a_{k,k}^{k-1}$, $\rho'^k := a_{k,k_s}^{k-1}$, and $F^{\rho'}$ be the frame matrix obtained using pivot sequence ρ' with (i,j)th entry as $a'^k_{i,j}$, then, for $i \neq k, j \notin \{k, k_s\}$, $a'^k_{i,j}$ can be equivalently obtained without roundoff error via the formula

$$a'^k_{i,j} = \frac{a_{i,j}^k \rho'^k - a_{i,k_s}^k a_{k,j}^{k-1}}{\rho^k} = \frac{a_{i,j}^k a_{k,k_s}^{k-1} - a_{i,k_s}^k a_{k,j}^{k-1}}{a_{k,k}^{k-1}}.$$

Dense REF LU Update

Theorem (prior dense case: APCPU shortcuts)

Let ρ' be the pivot sequence obtained by swapping column indices k and $k+1$ of feasible pivot sequence ρ. Then, for $i, j \geq k+2$ (i.e., frames $k+1$ to $n-1$),

$$a'^{\kappa}_{i,j} = -a^{\kappa}_{i,j}, \kappa = min(i,j) - 1.$$

In addition, for the $(k+1)$th frame-column entries with row index $i \geq k+1$,

$$a'^{k}_{i,k+1} = -a^{k}_{i,k+1}.$$

Dense REF LU Update

Theorem (prior dense case: APDPU shortcuts)

Let ρ' be obtained from swapping row and column indices k and $k+1$ of feasible pivot sequence ρ. For $i, j \geq k+2$ (i.e., frames $k+1$ to $n-1$),

$$a'^{\kappa}_{i,j} = a^{\kappa}_{i,j}, \kappa = min(i,j) - 1.$$

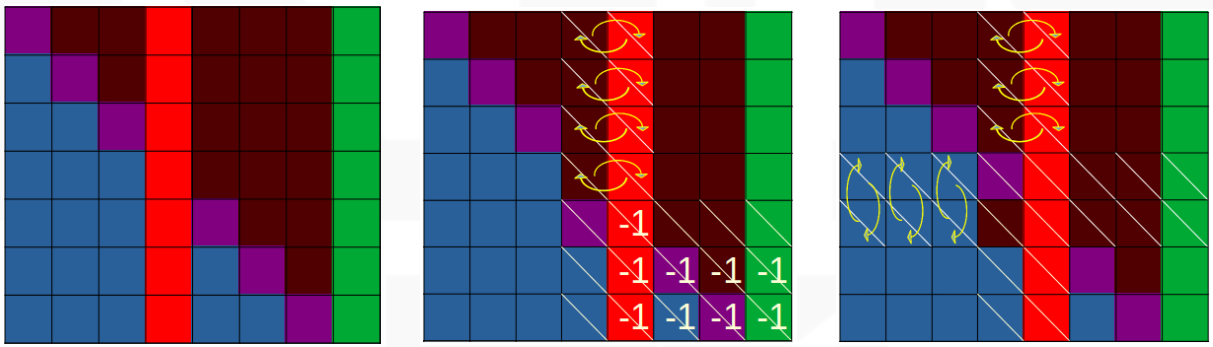
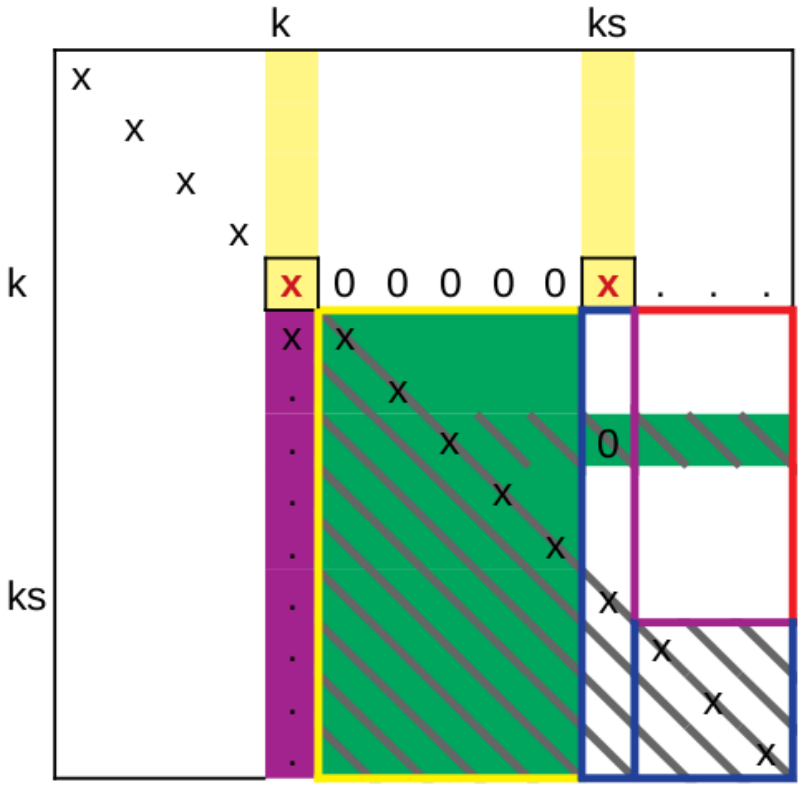


Figure: Left: original frame matrix F, Middle: updated F with APCPU, Right: updated F with APDPU.

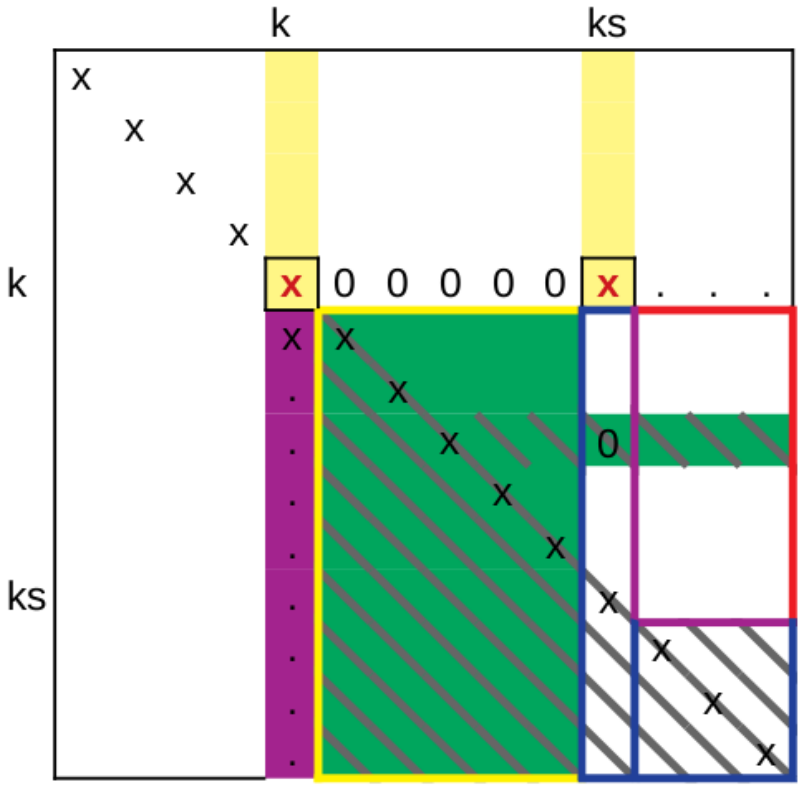
Column Permutation Pivot Update



$$\begin{bmatrix} a_{k,k}^{k-1} & 0 & 0 & a_{k,k_s}^{k-1} \\ \cdot & a_{i,j}^{k} & \cdots & a_{i,k_s}^{k} \\ \vdots & \cdot & \ddots & \vdots \end{bmatrix}$$

$$a'^{k}_{i,j} = \frac{a_{k,k_s}^{k-1}}{a_{k,k}^{k-1}} a_{i,j}^{k}$$

Column Permutation Pivot Update



$$\begin{bmatrix} a_{k,k}^{k-1} & 0 & a_{k,k_s}^{k-1} & a_{k,j}^{k-1} \\ \cdot & \cdots & a_{i,k_s}^{k} & a_{i,j}^{k} \\ \vdots & \ddots & \vdots & \cdot \end{bmatrix}$$

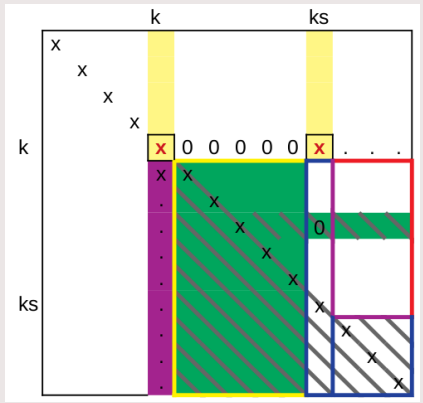
$$a'^{k}_{i,j} = \frac{a_{i,j}^{k} a_{k,k_s}^{k-1} - a_{i,k_s}^{k} a_{k,j}^{k-1}}{a_{k,k}^{k-1}}$$

- **Fill-in** could be added here.

Column Permutation Pivot Update

Theorem (new sparse case: RwSOP)

Let $a'^{\kappa}_{i,j}$, $a^{\kappa}_{i,j}, \kappa = min(i,j) - 1$ *be* (i,j)*th entry of original and updated frame matrix,respectively, then* $a'^{\kappa}_{i,j}, i \in [k+1, n], j \in [k+1, k_s - 1]$ *can be equivalently obtained without roundoff error via*

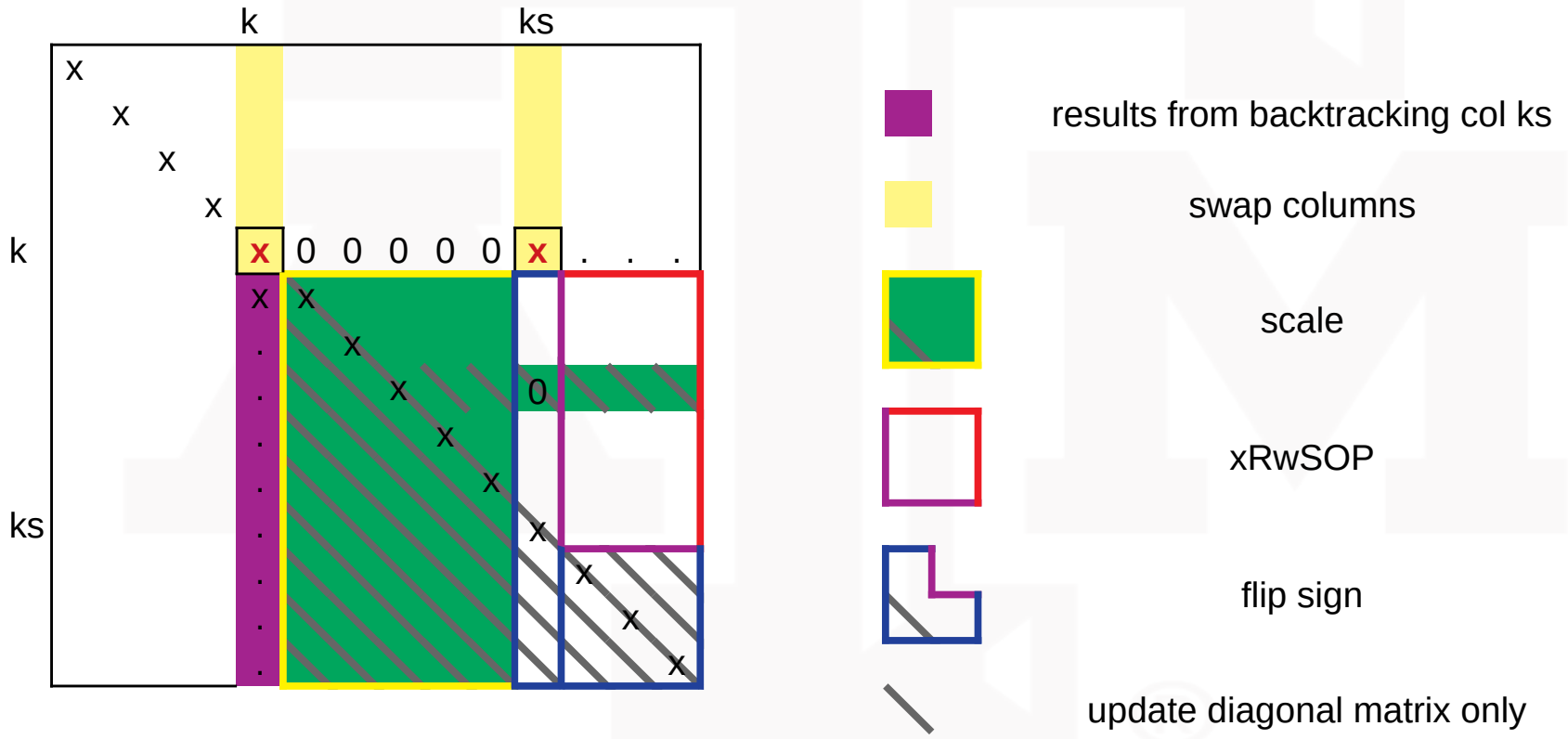


$$a'^{\kappa}_{i,j} = \frac{\rho'^{k}}{\rho^{k}} a^{\kappa}_{i,j} = \frac{a^{k-1}_{k,k_s}}{a^{k-1}_{k,k}} a^{\kappa}_{i,j}.$$

Moreover, $a'^{\kappa}_{i,j}, i \in [k+1, k_s], j \in [k_s+1, n+1]$ *can be equivalently obtained without roundoff error via*

$$a'^{\kappa}_{i,j} = \frac{a^{\kappa}_{i,j}\rho'^{k} - a^{\kappa}_{i,k_s} a^{k-1}_{k,j}}{\rho^{k}} = \frac{a^{\kappa}_{i,j} a^{k-1}_{k,k_s} - a^{\kappa}_{i,k_s} a^{k-1}_{k,j}}{a^{k-1}_{k,k}}$$

Column Permutation Pivot Update



Diagonal Permutation Pivot Update

Theorem (new sparse case: Backtracking in DwSOP)

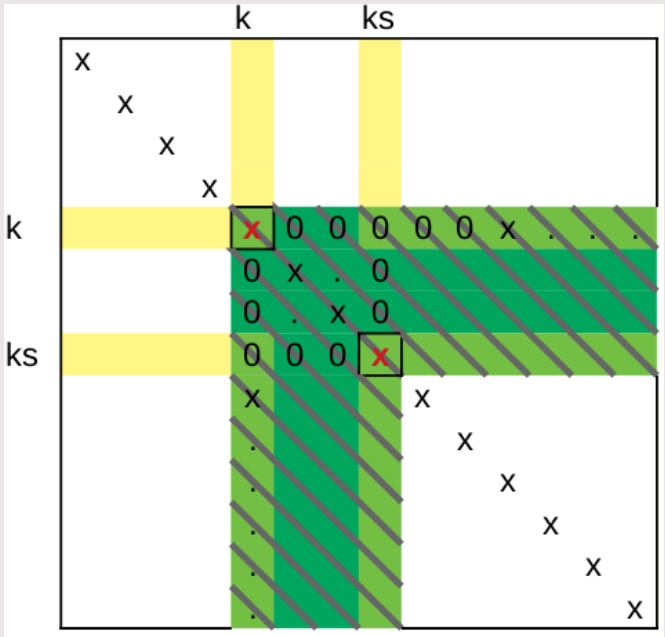
Let $a_{i,j}^{l}$ be (i,j)th entry of F^{ρ} in the lth IPGE iteration. Then, for $i,j \in [k+1,n]$, we have

$$a_{i,k_s}^{k-1} = \frac{\rho^{k-1}}{\rho^{\kappa}} a_{i,k_s}^{\kappa}, \kappa = min(i,k_s) - 1,$$

$$a_{k_s,j}^{k-1} = \frac{\rho^{k-1}}{\rho^{\kappa}} a_{k_s,j}^{\kappa}, \kappa = min(k_s,j) - 1.$$

Especially, for $i,j \in [k+1,k_s-1]$

$$a_{i,k_s}^{k-1} = a_{k_s,j}^{k-1} = 0.$$

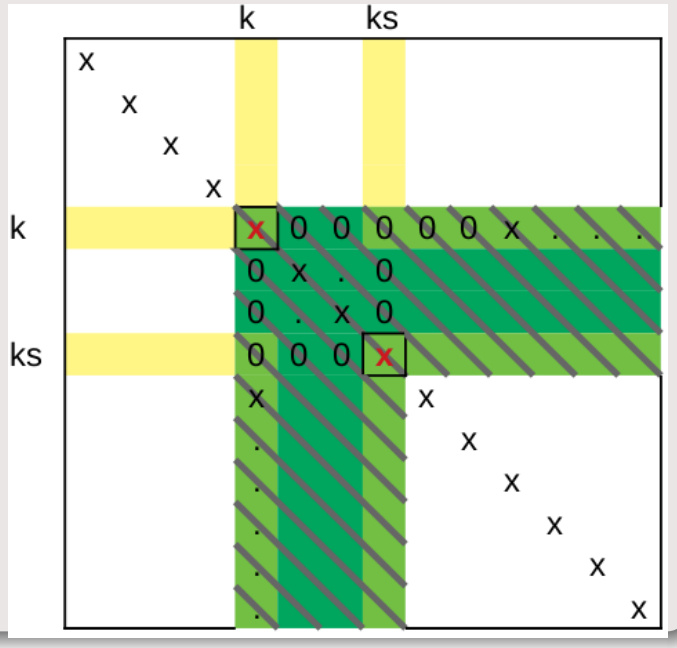


Diagonal Permutation Pivot Update

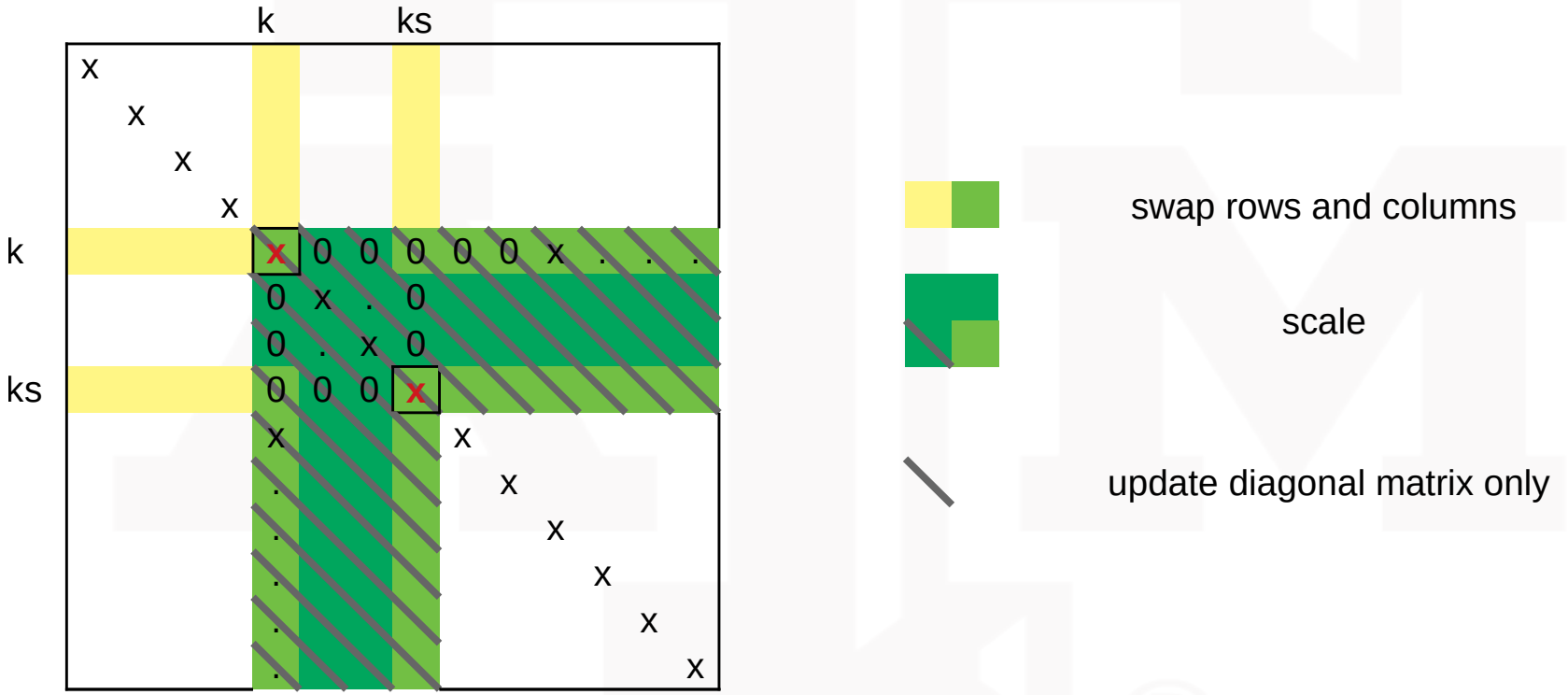
Theorem (new sparse case: DwSOP)

Let $a'^{\kappa}_{i,j}$, $a^{\kappa}_{i,j}$, $\kappa = min(i,j) - 1$ *be* (i,j)*th entry of* F^{ρ} *and* $F^{\rho'}$*, respectively, and define* $\mathbb{S} := \{(i,j) | i \in [k+1, n] \backslash \{k_s\}, j \in [k+1, n+1], min(i,j) \leq k_s - 1\}$*, then* $a'^{\kappa}_{i,j}$ *can be equivalently obtained without roundoff error via the formula*

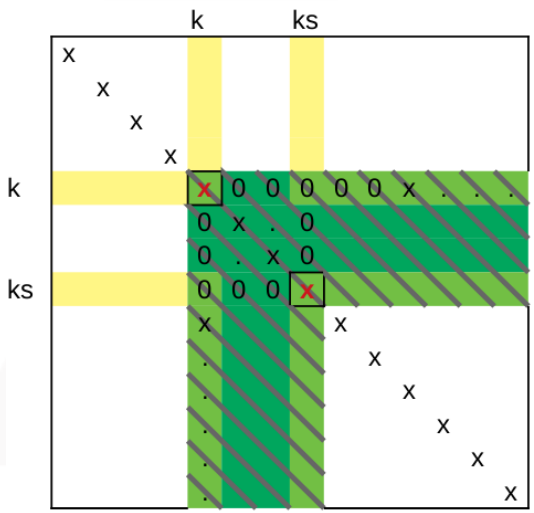
$$a'^{\kappa}_{i,j} = \frac{\rho'^{k}}{\rho^{k}} a^{\kappa}_{i,j} = \frac{a^{k-1}_{k_s,k_s}}{a^{k-1}_{k,k}} a^{\kappa}_{i,j}, (i,j) \in \mathbb{S}$$

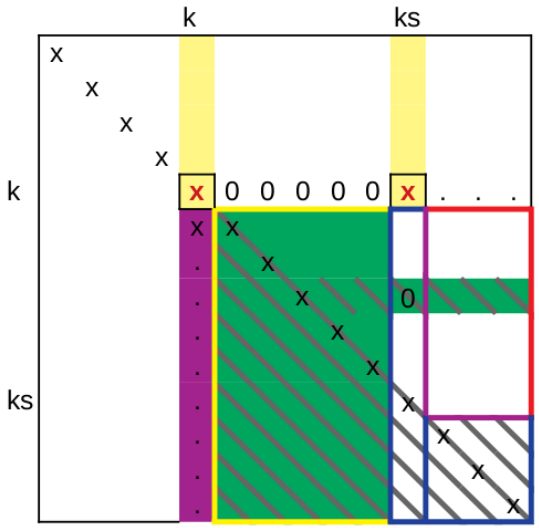


Diagonal Permutation Pivot Update



Sparse REF LU Update Algorithm

1. Solve y for $LD^{-1}y = v$ and append y as $n+1$-th column of U
2. while $k \leq n$
3. 

OR


4. end of while loop
5. swap columns n and $n+1$ of F, and delete column $n+1$ of F.
6. construct L, D and U from the resulted frame matrix F.

Conclusion

Contributions:

1. Rigid proof for several theorems as the basis of sparse REF LU update algorithm.
2. Propose the sparse REF LU update algorithm.
3. Successfully implement MATLAB version of proposed algorithm as a proof of concept.

Future work:

1. Implement a robust C library for the sparse REF LU update.