Deflation for the off-diagonal block in saddle-point systems

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Joint work with

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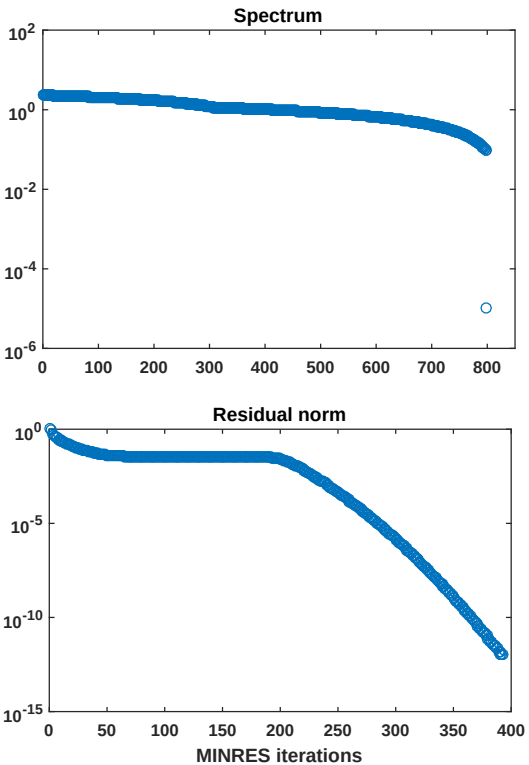
Deflation strategy goals

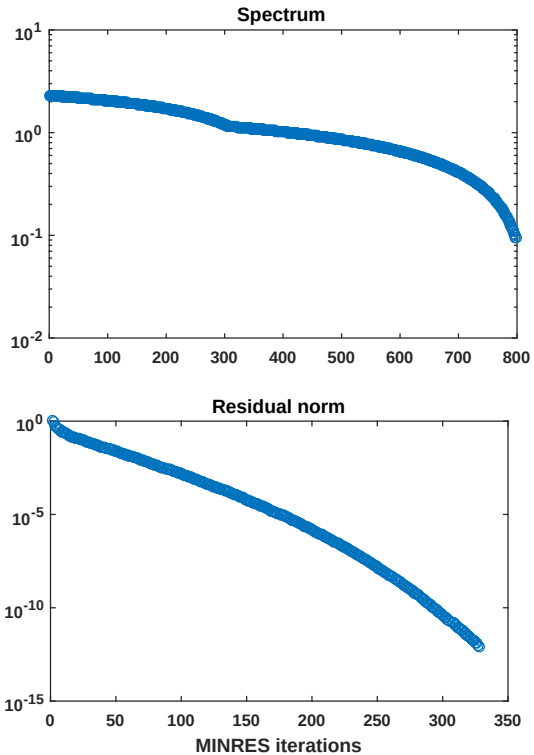
Before deflation*



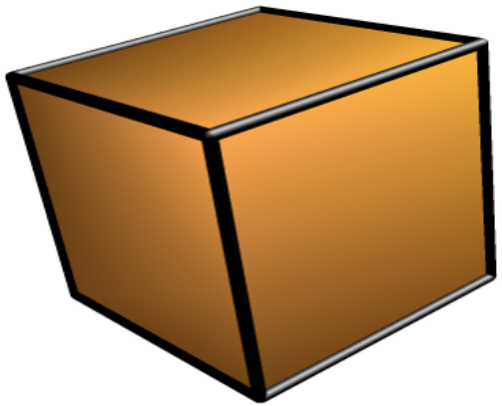
- Remove plateaus in Krylov subspace solver convergence
- Remove small, outlying eigenvalues
- Operate on the ill-conditiond, off-diagonal block in saddle-point systems
- Avoid working with the entire matrix

$$\begin{pmatrix} W & A \\ A^T & 0 \end{pmatrix}$$

After deflation



The deflation process



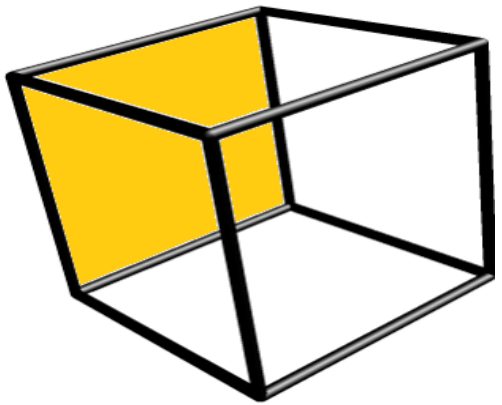
Range(A)

Deflation removes a part of the subspace represented by the system matrix.

U: Basis of the subspace to be deflated.

$P = I - UU^T$ Deflation operator

$M = U\left(U^T AU\right)^{-1} U^T$ Correction operator



Range(PA)

Original system: $Ax = b$

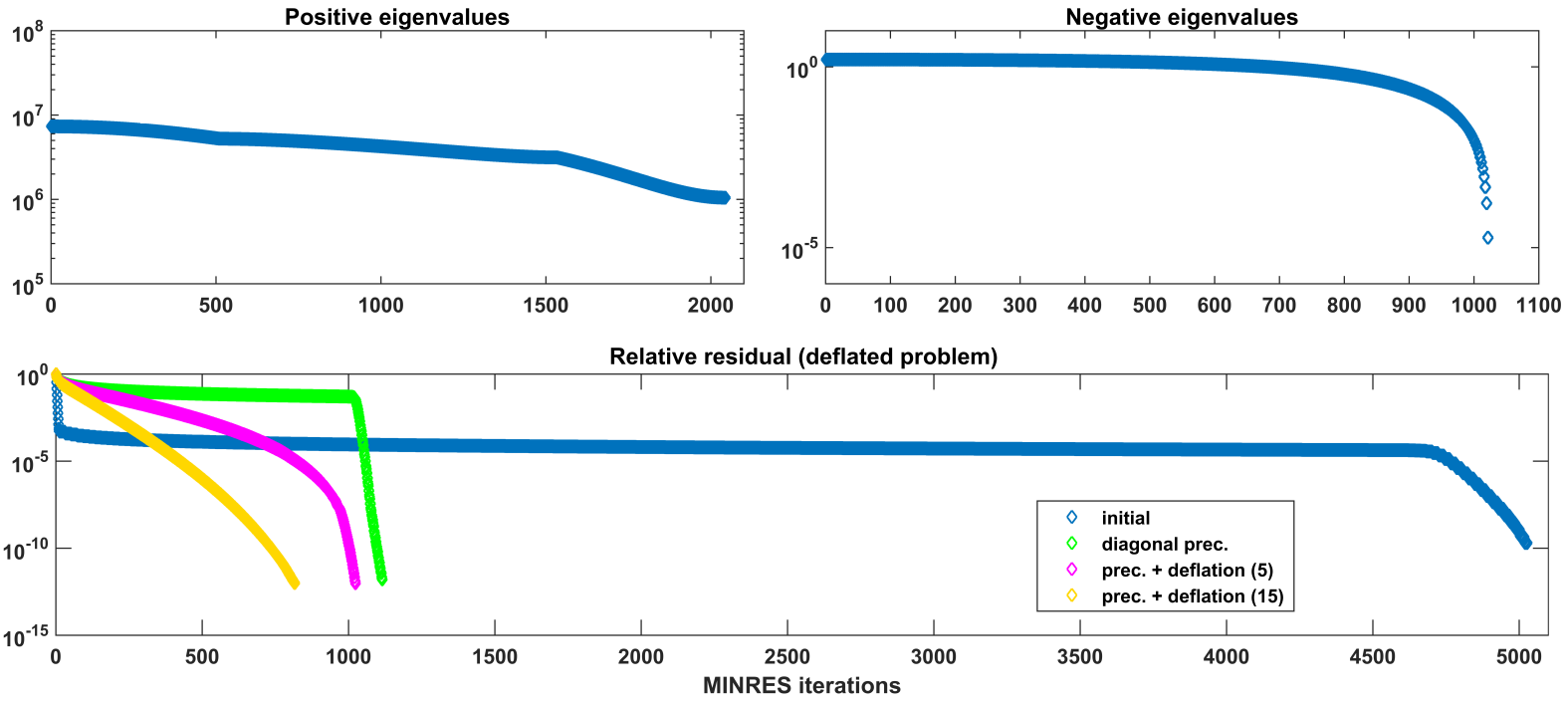
Deflated system: $PA\,\hat{x} = Pb$

Correction: $x = P\,\hat{x} + M\,b$

U can be represented by the *singular vectors* corrsponding to the *smallest singular values*, found using the Lanczos bidiagonalization and the Singular Value Decomposition.

MINRES and deflation

Figure: Spectrum of a saddle-point matrix from a Stokes flow problem in a 1D Channel and MINRES performance.



Deflation for saddle-point matrices and CRAIG

The CRAIG solver:

- operates block-wise,
- uses the generalized Golub-Kahan bidiagonalization,
- is equivalent to the Conjugate Gradient algorithm applied to the Schur complement,
- converges depending on the spectral distribution of the Schur complement.

Our deflation approach:

- operates block-wise,
- uses the generalized Golub-Kahan bidiagonalization to get the operators *P*, *Q* and *M*,
- implictly deflates the Schur complement,
- allows reusing block-specific solvers and structures.

Original system: $\begin{pmatrix} W & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} g \\ r \end{pmatrix}$

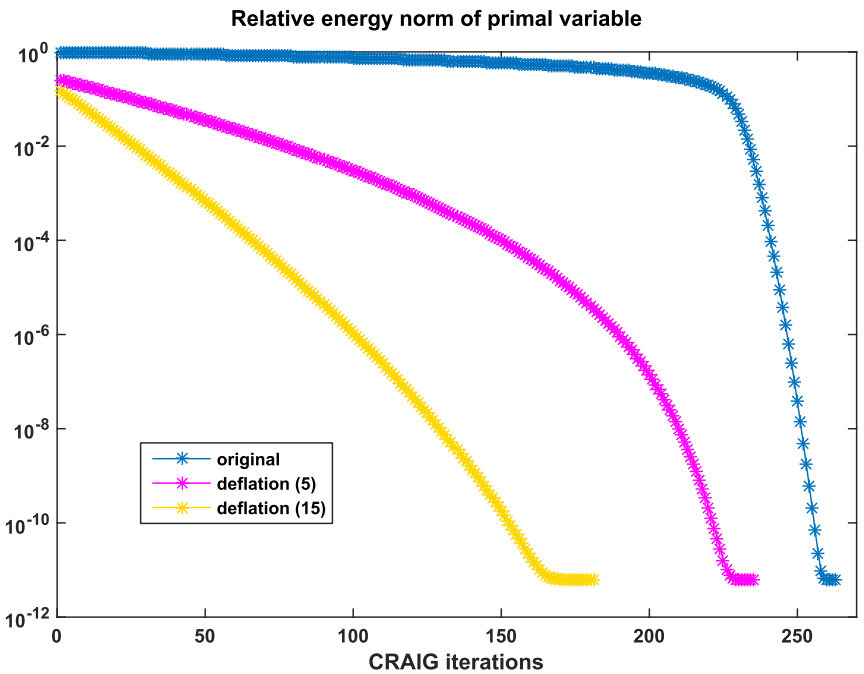
Deflated system: $\begin{pmatrix} W & AQ \\ Q^T A^T & 0 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} g \\ Q^T r \end{pmatrix}$

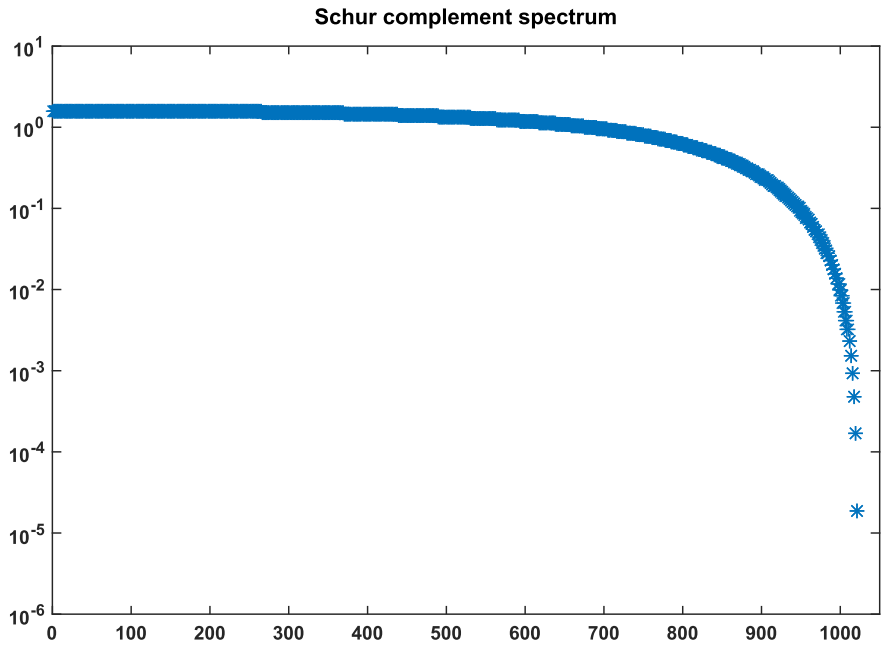
Correction: $u = P^T \hat{u} + M^T r$

$p = Q \hat{p} + M g - M W M^T r$

CRAIG and deflation

Figure: Spectrum of the Schur complement of a saddle-point matrix from a Stokes flow problem in a 1D Channel and CRAIG performance.





Further developments

- Use of Ritz vectors and values instead of exact spectral information
- Inexact inversion of the (1,1) block W
- Exploiting known problem properties in designing the deflation basis
- Cost analysis, possibly with multiple RHS solves

Thank you for your attention!

I will gladly answer related questions during the coffee break.

Selective bibliography

- Arioli, Mario. "Generalized Golub--Kahan bidiagonalization and stopping criteria." SIAM Journal on Matrix Analysis and Applications 34.2 (2013): 571-592.
- Barrett, Richard, et al. Templates for the solution of linear systems: building blocks for iterative methods. Society for Industrial and Applied Mathematics, 1994.
- Gaul, André, et al. "A framework for deflated and augmented Krylov subspace methods." SIAM Journal on Matrix Analysis and Applications 34.2 (2013): 495-518.
- * Example on slide 2 from :Simoncini, Valeria, and Daniel B. Szyld. "On the superlinear convergence of MINRES." Numerical Mathematics and Advanced Applications 2011. Springer, Berlin, Heidelberg, 2013. 733-740.
- Simoncini, Valeria, and Daniel B. Szyld. "Recent computational developments in Krylov subspace methods for linear systems." Numerical Linear Algebra with Applications 14.1 (2007): 1-59.