Toward robust, fast solutions of elliptic PDEs through HHO discretizations and multigrid solvers

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Sparse Days

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Outline

1. The Hybrid High-Order (HHO) method
2. Multigrid algorithm
3. Numerical results
4. Strengths and limitations

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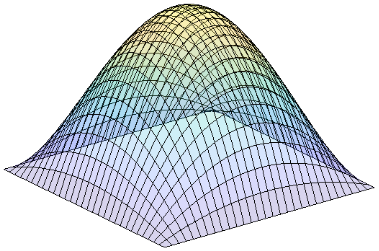
Model problem

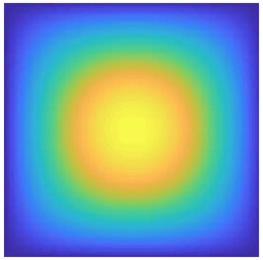
- Diffusion equation:

$$\begin{cases} -\nabla \cdot (\mathbf{K}\nabla u) = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

where $\mathbf{K}$ is uniformly elliptic and piecewise constant in Ω.

- Example of a 2D solution
  - homogeneous diffusion problem with constant source function:

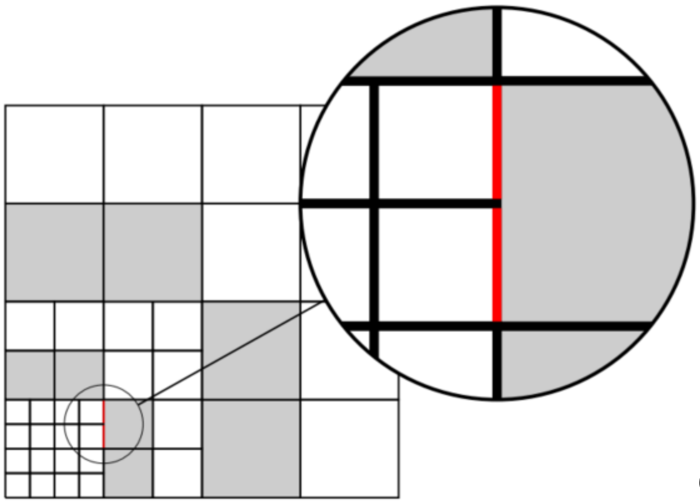




Hybrid High-Order (HHO)

HHO - Overview

- Applies to general polyhedral meshes
  - Easily approximates complex geometries through local mesh refinement
  - Non-conforming junctions perceived as coplanar faces

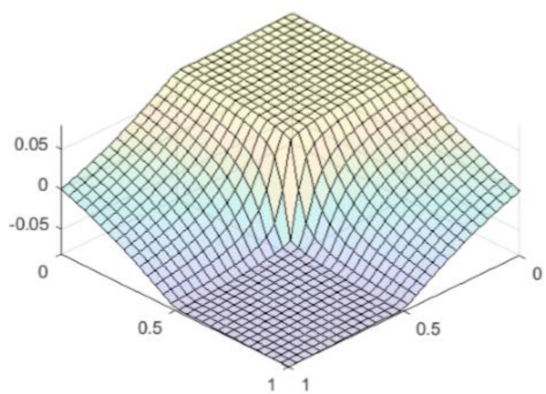


Gray elements are actually pentagons

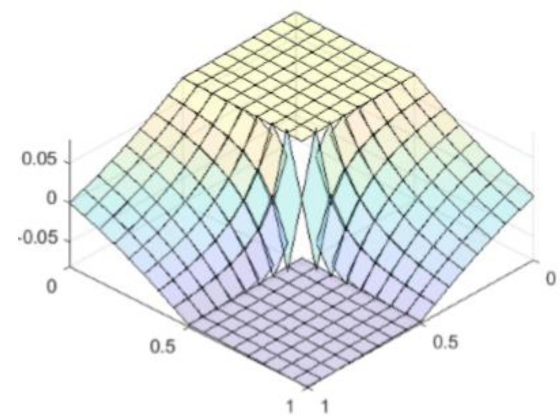
HHO - Overview

Discontinuous

- Suitable to approximate non-smooth solutions



Exact solution



HHO linear approximate solution

Hybridized

- Reduced number of degrees of freedom compared to standard Discontinuous Galerkin (DG) methods

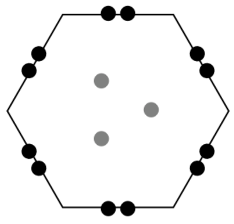
HHO - Overview

- Handles high order
  - The solution can be approximated by a polynomial of arbitrary degree $p \geq 1$
- Superconvergence
  - If $u \in H^{p+1}(\Omega)$, convergence in $\mathcal{O}(h^{p+1})$ in L^2-norm
- More efficient than regular HDG
  - Can be seen as a subclass of Hybridized Discontinuous Galerkin (HDG) methods
  - 1 extra order of convergence compared to HDG, because of a special stabilization term

HHO – Degrees of Freedom

Given an admissible mesh $(\mathcal{T}_h, \mathcal{F}_h)$ and a polynomial order $k \in \mathbb{N}$:

- the DoFs are located in cells and on faces, defining polynomials of degree k
- Space of unknowns local to $T \in \mathcal{T}_h$:

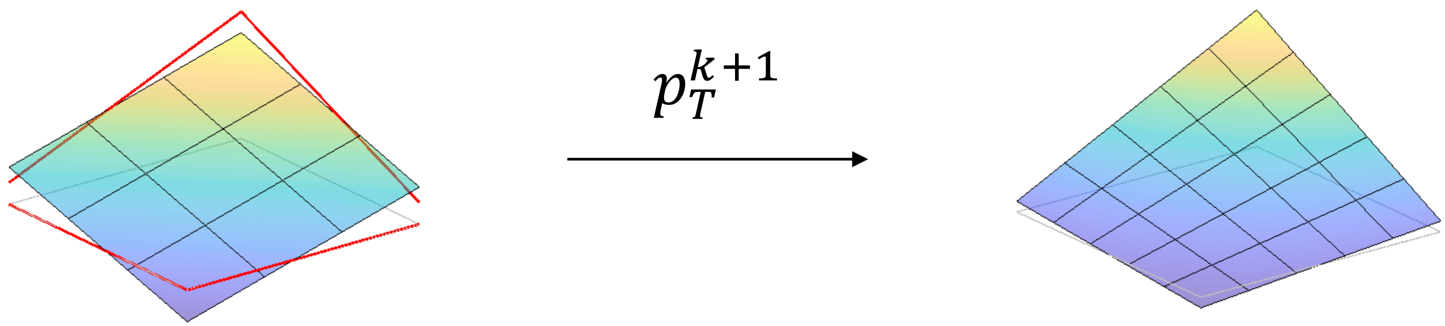


$$\left\{ \left(v_T, (v_F)_{F \in \mathcal{F}_T} \right) \;\middle|\; \begin{array}{l} v_T \in \mathbb{P}^k(T) \\ v_F \in \mathbb{P}^k(F) \quad \forall F \in \mathcal{F}_T \end{array} \right\}$$

- $\forall v \in L^2(\Omega)$, $$\begin{array}{l} v_T \coloneqq \pi_T^k v \\ v_F \coloneqq \pi_F^k v \quad \forall F \in \mathcal{F}_T \end{array}$$
  where $\pi_T^k v$ (resp. $\pi_F^k v$) is the L^2-orthogonal projector onto $\mathbb{P}^k(T)$ (resp. $\mathbb{P}^k(F)$).

HHO – Higher-order reconstruction

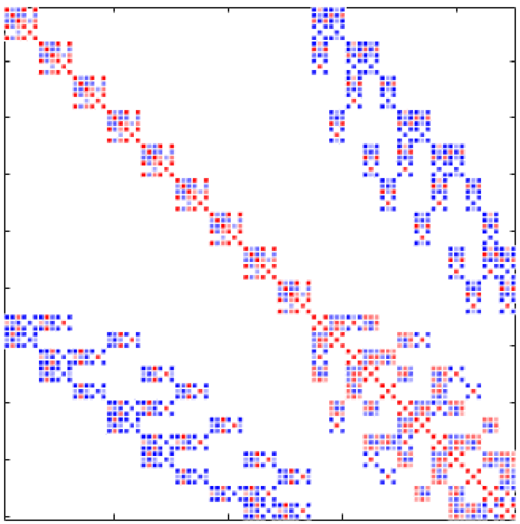
- The local reconstruction operator p_T^{k+1} allows to gain one order of approximation.



$\mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_T)$ $\qquad\qquad$ $\mathbb{P}^{k+1}(T)$

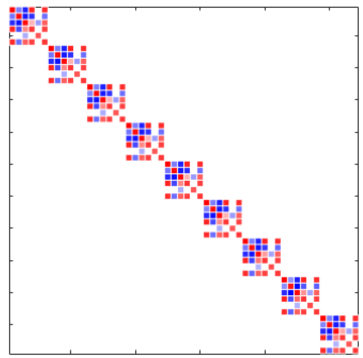
- Based only on an **integration by parts** and the enforcement of a constraint imposing that **the cell-based approximations of degree k and $k+1$ have the same average value**.

HHO - Static condensation

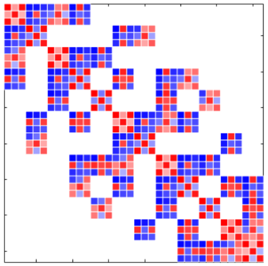


Global matrix

-1

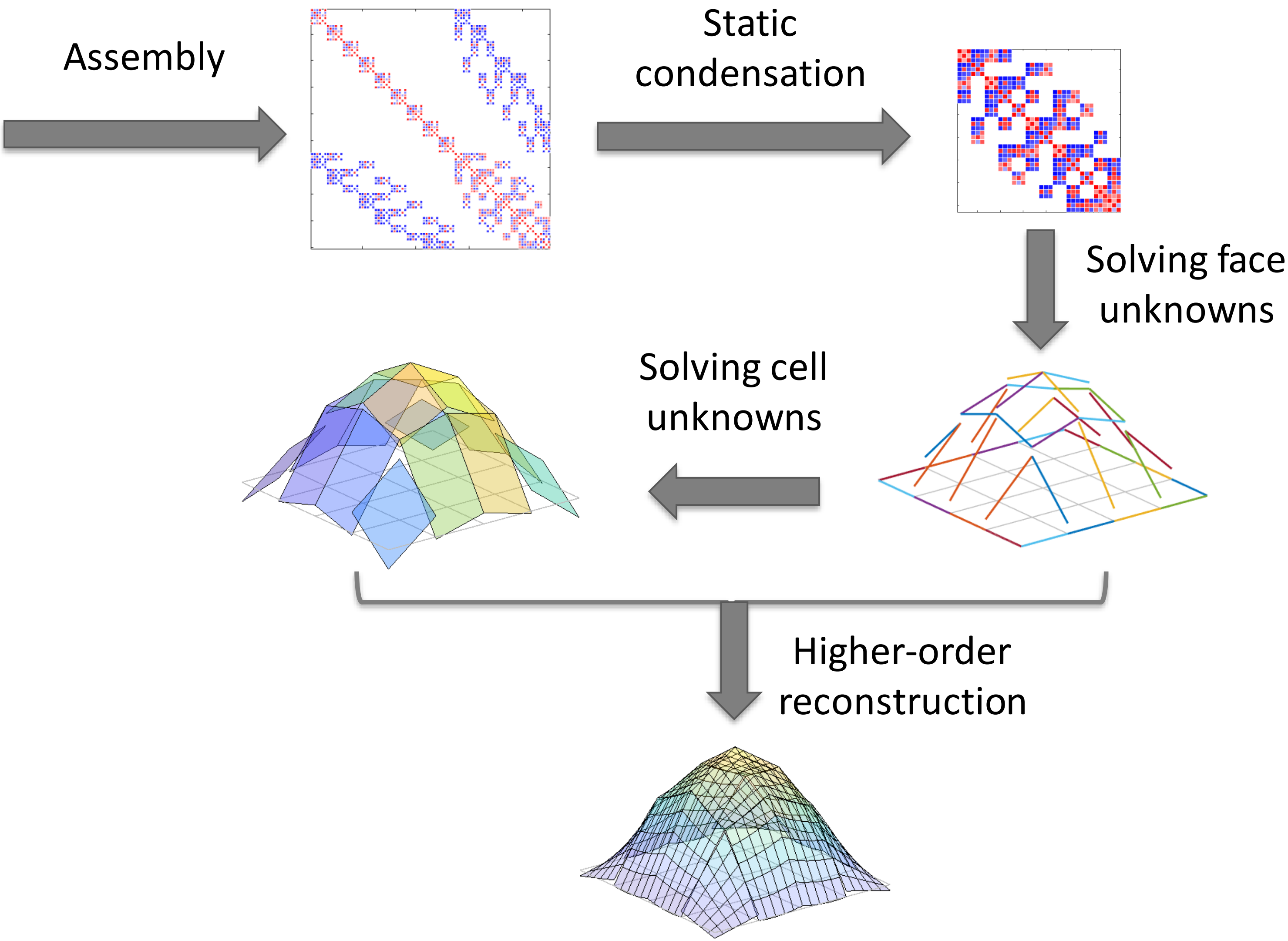


Local elimination
of cell unknowns



Schur complement

HHO - Summary

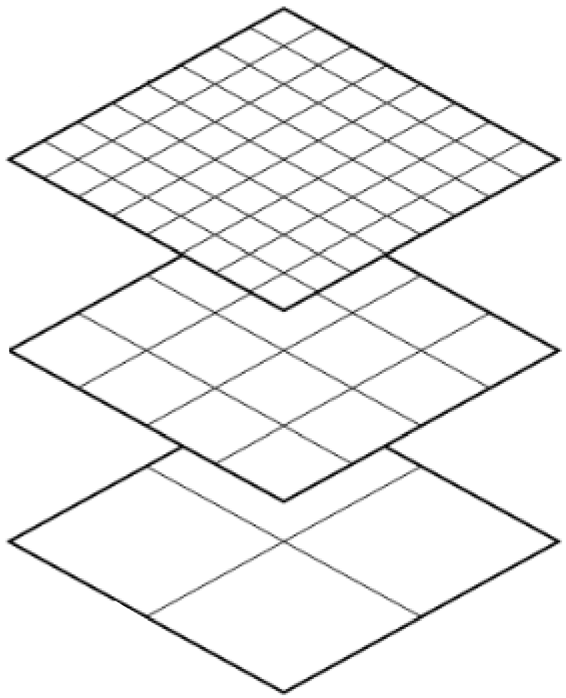


Assembly
Static condensation
Solving face unknowns
Solving cell unknowns
Higher-order reconstruction

Multigrid algorithm

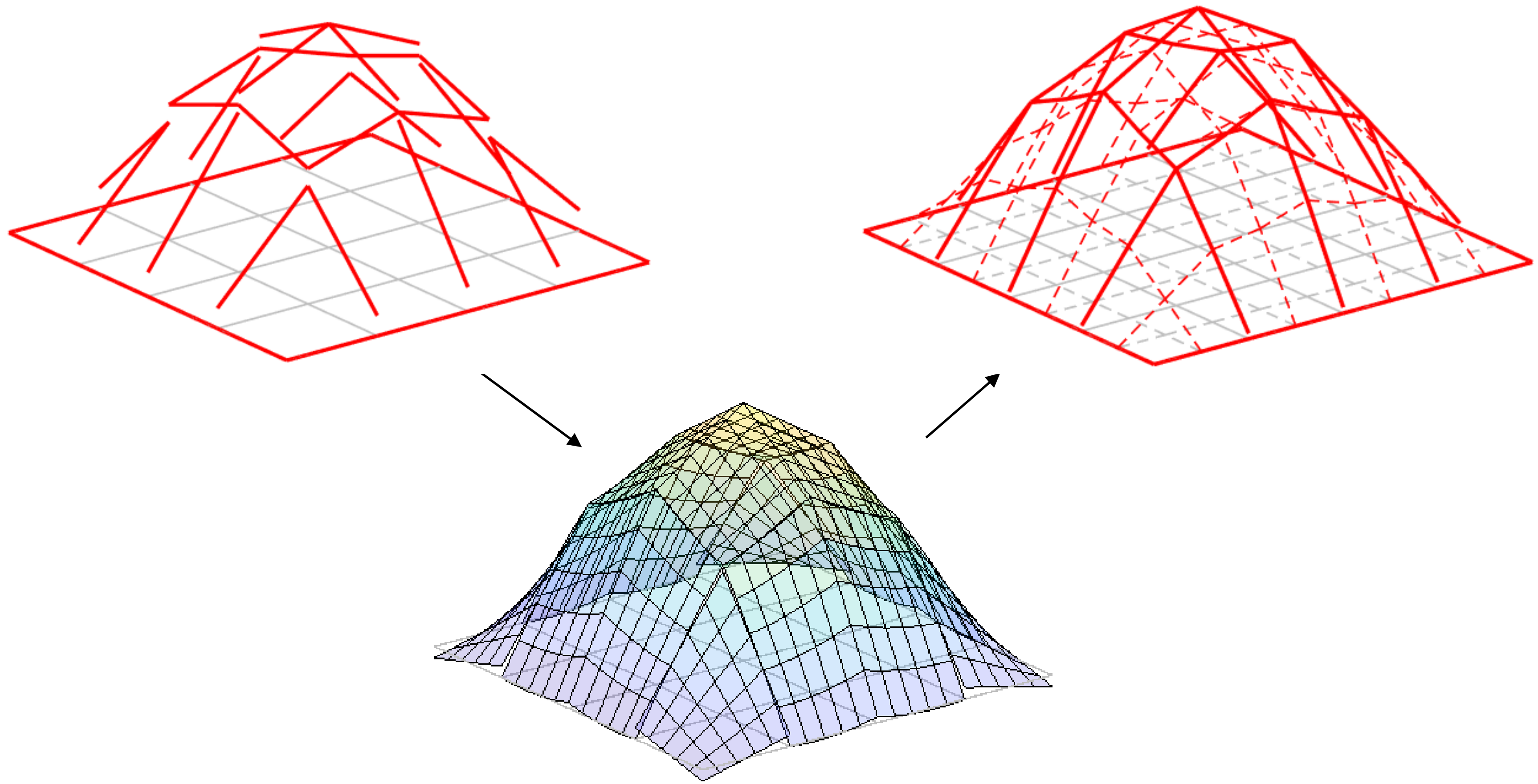
Multigrid

Prolongation



Restriction

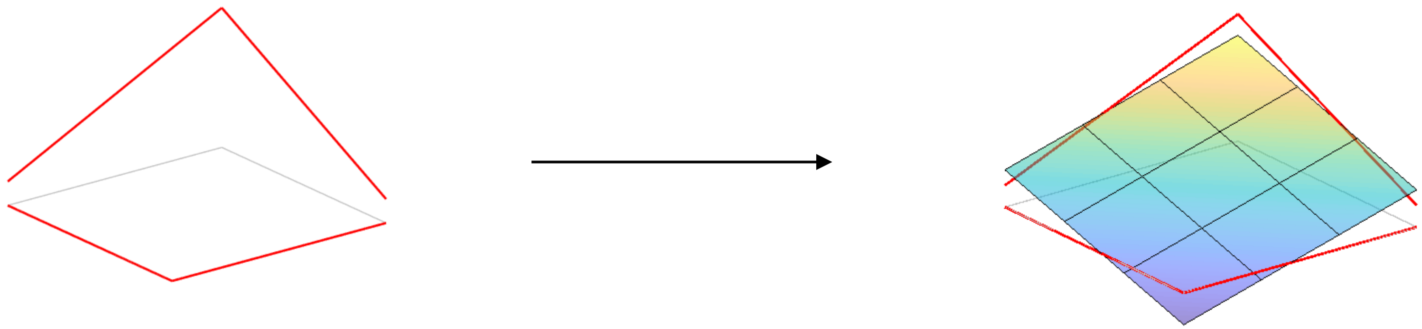
Prolongation



Intermediary step: interior reconstruction

From the faces to the cell

Step 1: reconstruction of degree k



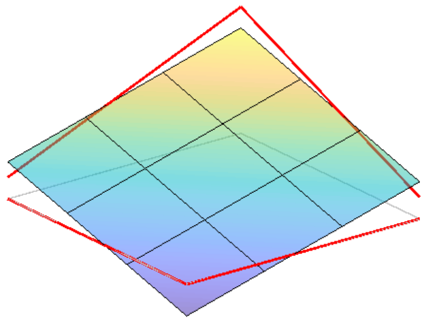
$v_{\partial T} \in \mathbb{P}^k(\mathcal{F}_T)$ $\qquad$ $v_T \in \mathbb{P}^k(T)$

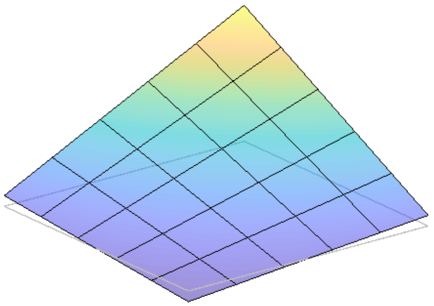
The static condensation is reversed by the solution of the local problem:

$$A_T := \begin{pmatrix} A_{TT} & A_{T\mathcal{F}_T} \\ A_{\mathcal{F}_T T} & A_{\mathcal{F}_T\mathcal{F}_T} \end{pmatrix} \qquad v_T := -A_{TT}^{-1} A_{T\mathcal{F}_T} v_{\partial \mathrm{T}}$$

From the faces to the cell

Step 2: reconstruction of degree $k+1$



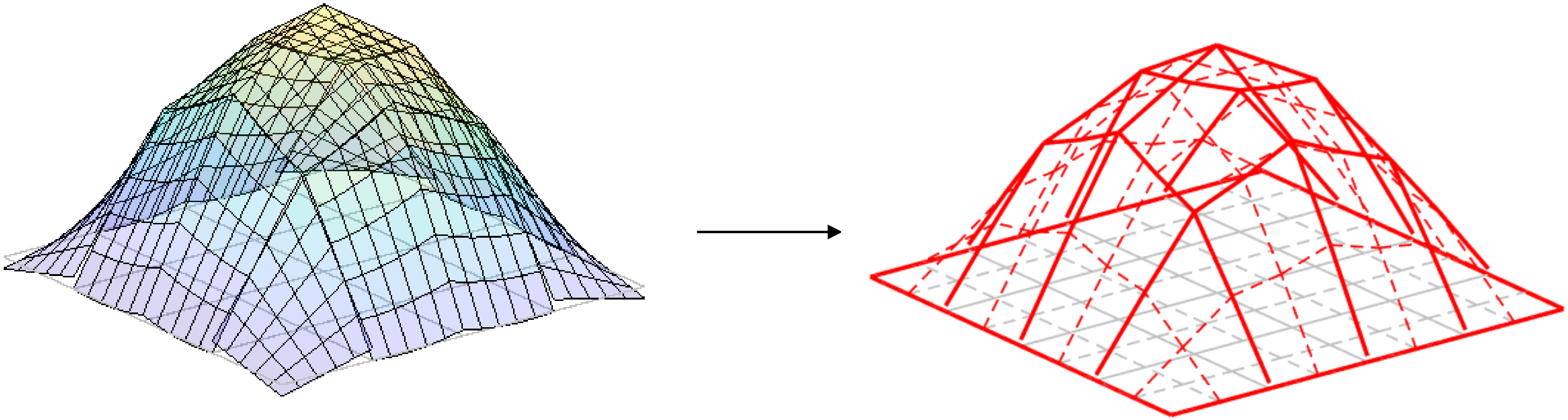


$$(v_T, v_{\partial T}) \in \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_T) \qquad v_T^{k+1} \in \mathbb{P}^{k+1}(T)$$

The local reconstruction operator p_T^{k+1} is used:

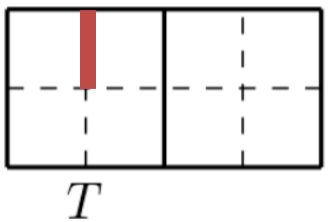
$$v_T^{k+1} \coloneqq p_T^{k+1}(v_T, v_{\partial T})$$

From coarse cells to fine faces



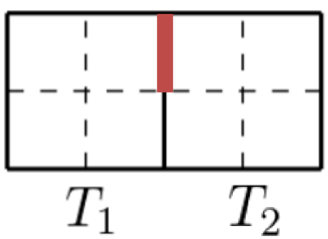
From coarse cells to fine faces

- Interior to 1 coarse element:



$$v_F := \pi_F^k v_T^{k+1}$$

- At the interface between 2 coarse elements:



$$v_F := \sum_{i=1}^{2} w_{T_iF} \pi_F^k v_{T_i}^{k+1}$$

with $\sum_{i=1}^{2} w_{T_iF} = 1$

Multigrid components

- The prolongation P is defined as the composition of the two previous operators.
- Restriction: $R := P^T$
- Coarse grid operators: discretized operator on the coarse meshes or Galerkin construction.
- Smoother: block-version of standard fixed-point iterative methods (block size = number of DOFs per face).

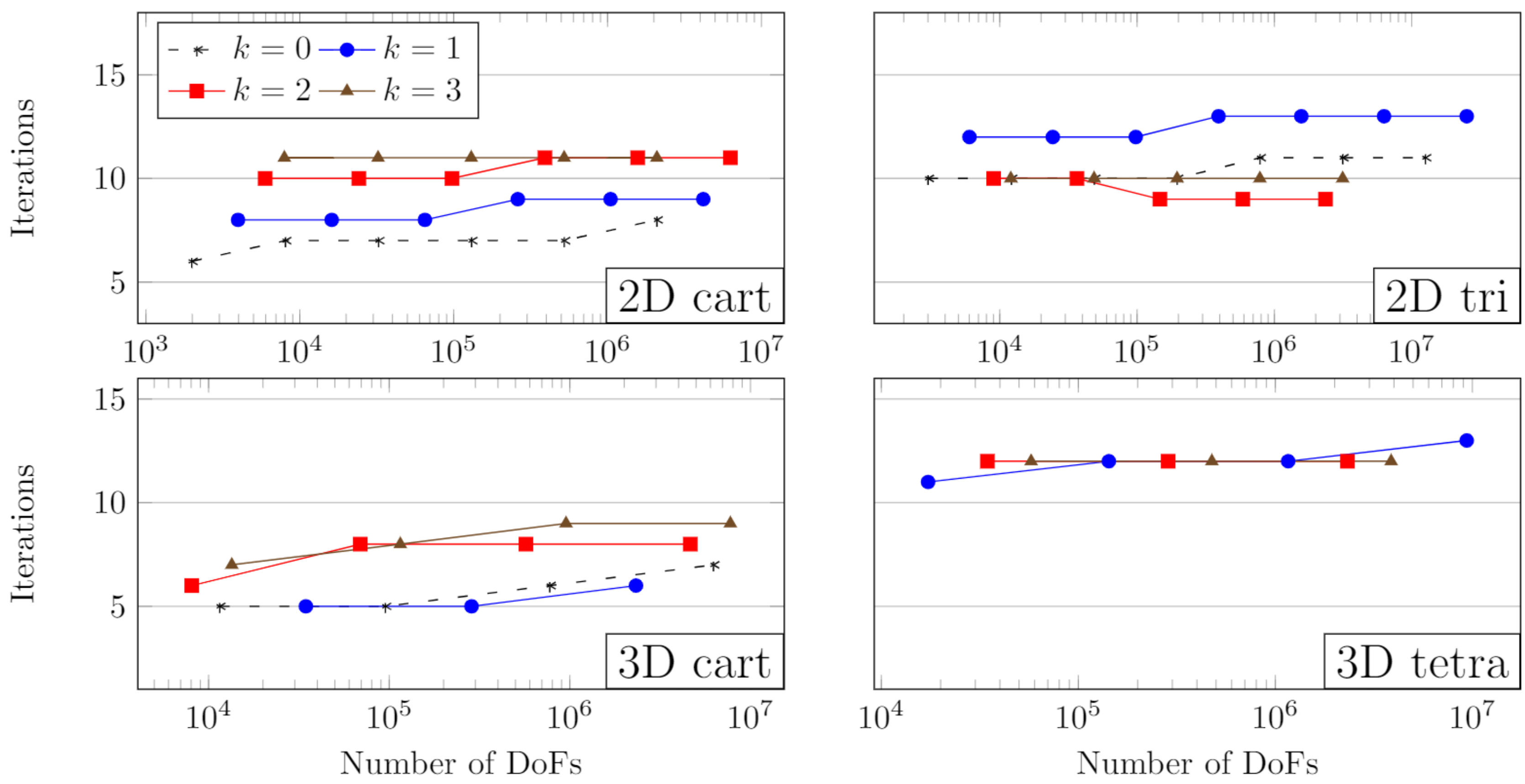
Numerical results

Testing conditions

- Smoother: block-Gauss-Seidel in presmoothing, block-Gauss-Seidel in the reverse order as post-smoothing.
- Discretized operator on the coarse meshes
- Stopping criterion: $\frac{||r||}{||b||} < 10^{-8}$
- Coarsening strategy: standard coarsening on Cartesian/triangular grids
- Analytical solution: sine function

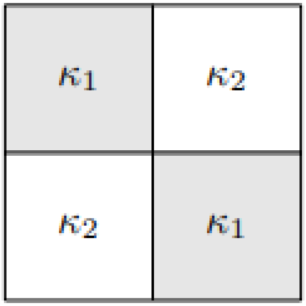
Homogeneous diffusion

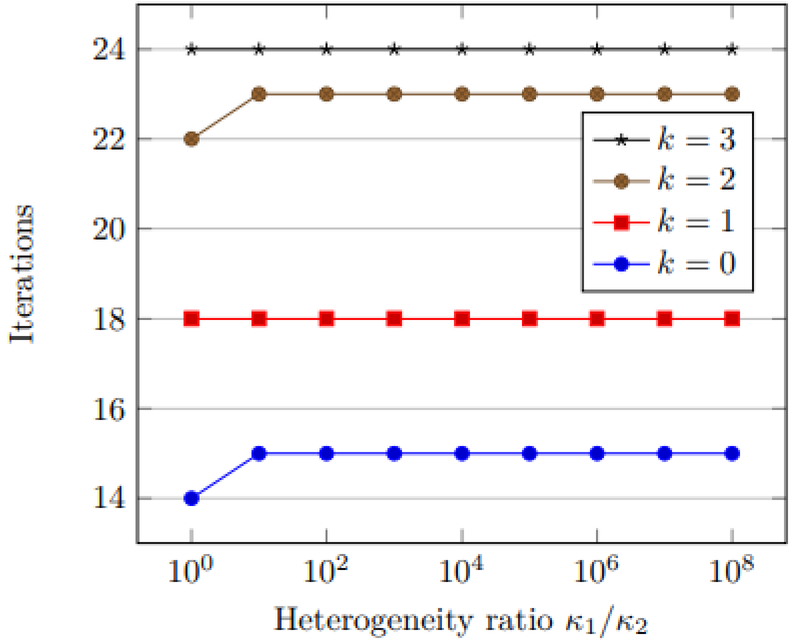
Multigrid as a solver, with the V(0,3) cycle in 2D (square) and V(0,6) cycle in 3D (cube)



Heterogeneous diffusion

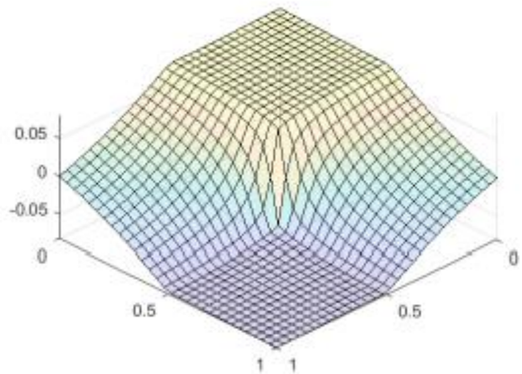
Square domain partitioned in 4 quadrants, V(1,1)



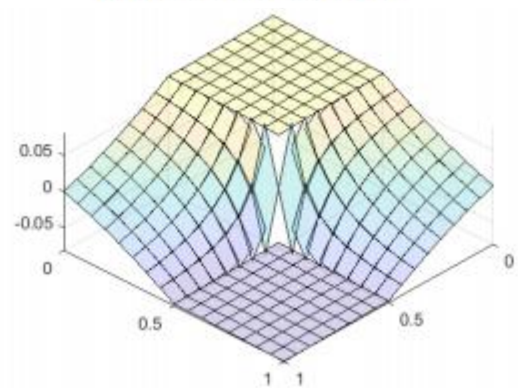


Heterogeneous diffusion

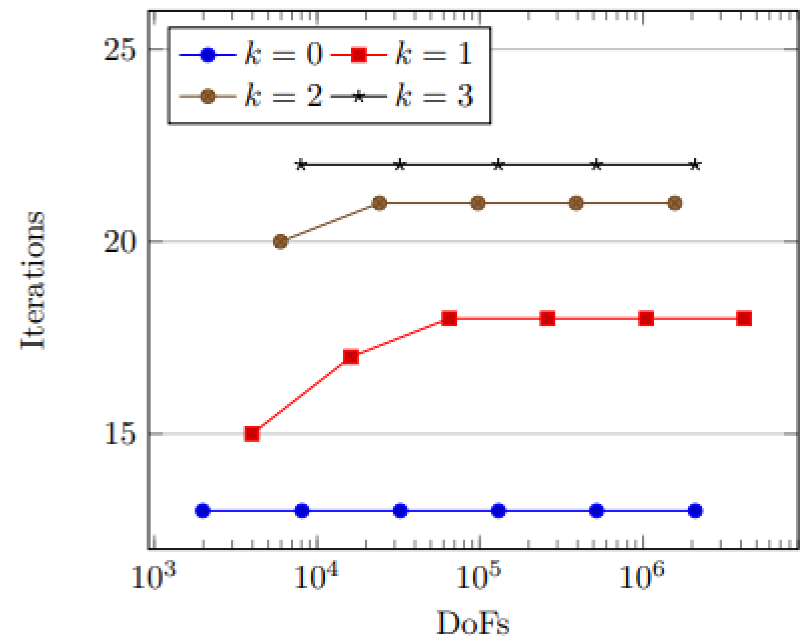
(Benchmark) Kellogg problem



(a) Analytical solution.



(b) HHO($k = 0$) $N = 16$ approximate solution.



Qualitative analysis

Strengths of the method

- Performance
  - fast
  - scalable
  - robust to heterogeneity
  - same properties in higher orders
- Higher orders natively managed
  - No need for an additional p-multigrid on top
- No restriction on the mesh structure
  - Applicable to unstructured polyhedral meshes

Limitations

- No obvious admissible coarsening strategy for polyhedral meshes
  - Faces must also be coarsened so the smoother can work on the low frequencies
- The algorithm seems very sensitive to the mesh quality
  - Which is a problem in 3D, because conserving the mesh quality upon refinement or coarsening becomes a difficult problem

Thank you

References:

[1] D. A. Di Pietro and J. Droniou, *The Hybrid High-Order method for polytopal meshes*, no. 19 in Modeling, Simulation and Application, Springer International Publishing, 2020.
[2] P. Matalon, D. A. Di Pietro, F. Hülsemann, P. Mycek, D. Ruiz, U. Rüde, *An h-multigrid method for Hybrid High-Order discretizations*, preprint available on HAL, 2020.