Stability of Certain Canonical Forms under Small Perturbations

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Small perturbation problems

- We consider $A_0 \in \mathbb{C}^{n\times n}$ and $A \in \mathbb{C}^{n\times n}$, with $\| A - A_0 \|$ be sufficiently small.

- What happens with
  - eigenvalues?

    We discuss Lipschitz type bounds
    - $|\lambda_0 - \lambda_i| \leq K \| A - A_0 \|$

    Hölder type bounds
    - $|\lambda_0 - \lambda_i| \leq K \| A - A_0 \|^{\frac{1}{n}}$
  - Jordan structure?

  - Jordan bases? and more...

Example:Companion Matrices

$$\det\left(xI - \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}\right) = x^3 + a_2 x^2 + a_1 x + a_0$$

$\det\left(xI - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = x^3$	*one eigenvalue* $\sigma(A_0) = \{\lambda_0 = 0\}$
$\det\left(xI - \begin{bmatrix} 0 & 0 & \varepsilon \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = x^3 - \varepsilon$	*three eigenvalues* $\sigma(A) = \{\lambda_1 = \varepsilon^{\frac{1}{3}}, \lambda_2 = \varepsilon^{\frac{1}{3}} e^{\frac{2i\pi}{3}}, \lambda_3 = \varepsilon^{\frac{1}{3}} e^{-\frac{2i\pi}{3}}\}$

- We have $\| \lambda_0 - \lambda_1 \| = \| A - A_0 \|^{\frac{1}{3}}$
- Eigenvalues split $\Rightarrow$ Hölder type bound

Companion Matrices

- Do we always have Lipschitz-type bounds? No

- The bound is not of Lipschitz type
$$|\lambda_0 - \lambda_i| \leq K|0 - \varepsilon^{\frac{1}{3}}| = K \parallel A - A_0 \parallel^{\frac{1}{3}}$$
when the eigenvalues split.

- Are there special cases when we have Lipshitz-type bound? Yes
$$|\lambda_0 - \lambda_1| \leq K \parallel A - A_0 \parallel$$

Single eigenvalues that do not split

- If we have $\sigma(A_0) = \{\lambda_0\}$ and $\sigma(A) = \{\lambda\}$, then we have the Lipschitz type

$$|\lambda_0 - \lambda| = \frac{1}{n}|tr(A_0) - tr(A)| \leq K \parallel A - A_0 \parallel$$

  No Splitting $\Rightarrow$ Lipschitz bound

- What happens with
  - eigenvalues ✓
  - Jordan structure
  - Jordan bases

Permutation of the Jordan Structure

- A_0 has only one eigenvalue 0.

$$A_0 = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & {\color{red}0} \\ \hline 0 & 0 & 0 \end{array}\right]$$

- A has only one eigenvalue 0 as well.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & {\color{red}\varepsilon} \\ 0 & 0 & 0 \end{bmatrix}$$

- However
  - A_0 has two Jordan blocks
  - A has only one
- Jordan structure is unstable!

What happens with the Jordan bases if the eigenvalues do not split and the Jordan structure does not change at all?

- What happens with
  - eigenvalues ✓
  - Jordan structure ✓
  - Jordan bases?

Theorem(2008, T.Bella, V.Olshevsky, U.Prasad)

Let us assume that

- no splitting eigenvalues
- no change of the Jordan structure

	A_0	A
Eigenvalues	λ_0	λ
Jordan bases	$f_0^{(1)} \quad \dots \quad f_{(m_1-1)}^{(1)}$ $\vdots \qquad\qquad \vdots$ $f_0^{(k)} \quad \dots \quad f_{(m_k-1)}^{(k)}$	$g_0^{(1)} \quad \dots \quad g_{(m_k-1)}^{(1)}$ $\vdots \qquad\qquad \vdots$ $g_0^{(k)} \quad \dots \quad g_{(m_k-1)}^{(k)}$

The following Lipschitz bound holds.

$$\| g_r^{(k)} - f_r^{(k)} \| \leq K \| A - A_0 \| \qquad k = 1, 2, \dots, d (\text{for some } d)$$

The constant $K > 0$ depends on fixed A_0 and fixed $\{f_r^{(k)}\}_{r=0}^{m_k(A_0,\lambda_0)-1}$ only.

Motivation

- Bella, Prasad and Olshevsky (2008) proved the Lipschitz stability of the so-called flipped orthogonal (FO) Jordan bases of complex H-selfadjoint matrices under small perturbations.
- Leiba Rodman asked whether the real version of this result holds true.

  It turns out that in order to answer Rodman's question, we need to introduce a new canonical form for H-selfadjoint matrices, essentially a new Jordan basis.

Indefinite Inner Product

$[.,.]$ is an indefinite inner product in $\mathbb{C}^n$ if the following holds:

- it is linear in the first argument: $[ax+by,z]=a[x,z]+b[y,z]$ for all $x,y,z\in\mathbb{C}^n$ and all $a,b\in\mathbb{C}$;
- it is antisymmetric: $[x,y]=\overline{[y,x]}$ for all $x,y\in\mathbb{C}^n$;
- it is non-degenerate: If $[x,y]=0$ for all $y\in\mathbb{C}^n$ then $x=0$.

$$[x,y]_H=\langle Hx,y\rangle=y^*Hx$$

What if H is not I? (Example for Real Eigenvalues)

Let

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix},$$

where $\varepsilon = \pm 1$, then $PJ = J^* P$.

	e_1	e_2
e_1	$[e_1, e_1]_P$	$[e_1, e_2]_P$
e_2	$[e_2, e_1]_P$	$[e_2, e_2]_P$

vs. $P = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}$

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vs. $P = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}$

What if H is not I? Part II (Example for Complex Eigenvalues)

Let

$$J = \left[\begin{array}{cc|cc} 1+i & 1 & & \\ 0 & 1+i & & \\ \hline & & 1-i & 1 \\ & & 0 & 1-i \end{array}\right] \text{ and } P = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right],$$

then $PJ = J^* P$ shows that corresponding eigenvalues come with its conjugate.

Such matrix P is called a **sip** (standard involuntary permutation) matrix.

Flipped Orthogonality

	e_1	e_2	e_3	e_4
e_1	$[e_1, e_1]_P$	$[e_1, e_2]_P$	$[e_1, e_3]_P$	$[e_1, e_4]_P$
e_2	$[e_2, e_1]_P$	$[e_2, e_2]_P$	$[e_2, e_3]_P$	$[e_2, e_4]_P$
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vs. $P = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$

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Affiliation relation

Two pairs (A, H) and (J, P) are called affiliated if

$$T^{-1}AT = J \quad \text{and} \quad T^{*}HT = P.$$

Matrix T is called the affiliation matrix.
Let us consider the case when J is a Jordan form of A:

$$J = \underbrace{J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha)}_{\text{real eigenvalues}} \oplus \underbrace{\hat{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \hat{J}(\lambda_\beta)}_{\text{non-real eigenvalues}}$$

$$\downarrow$$

$$\hat{J}(\lambda_k) = \begin{bmatrix} J(\lambda_k) & 0 \\ 0 & J(\overline{\lambda}_k) \end{bmatrix}$$

Canonical Basis for H-selfadjoint Matrices

Let us partition T correspondingly

$$T = [\,\underbrace{T_1 \ldots T_\alpha}_{\text{real } \lambda_k\text{'s}} \mid \underbrace{T_{\alpha+1} \ldots T_\beta}_{\text{non-real } \lambda_k\text{'s}}\,].$$

Moreover, the columns of

$$T_k = [\underbrace{g_{0,k}, \ldots, g_{p_k-1,k}}_{\lambda_k} \mid \underbrace{h_{0,k}, \ldots, h_{p_k-1,k}}_{\overline{\lambda}_k}]$$

for $k = \alpha + 1, \ldots, \beta$ form two Jordan chains, corresponding to λ_k and $\overline{\lambda}_k$ respectively.

Flipped-Orthogonal Basis

Recall $T^*HT = P$, where

$$P = P_1 \oplus \cdots \oplus P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P_\beta.$$

Then from structure of matrix P we can have the following orthogonality relation for any vectors g and h from the Jordan basis

$$g^* Hh = e_k^* (T^* HT) e_j = e_k^* P e_j$$

for some k, j.
Jordan bases, having the above property, are called flipped-orthogonal (FO).

Flipped-Orthogonal Basis

The form (J, P) is called the Weierstrass form of (A, H), where J is the Jordan canonical form and P is a sip matrix, and (J, P) is affiliated with (A, H), i.e. the corresponding Jordan basis is flipped-orthogonal .

Example

$$J = \left[\begin{array}{cc|cc|cc} 2 & 1 & & & & \\ 0 & 2 & & & & \\ \hline & & 1+i & 1 & & \\ & & 0 & 1+i & & \\ \hline & & & & 1-i & 1 \\ & & & & 0 & 1-i \end{array}\right] \text{ and } P = \left[\begin{array}{cc|cc|cc} 0 & \varepsilon & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right],$$

γ-Conjugate Symmetric Basis

- If $A \in \mathbb{R}^{n\times n}$ and there exists an invertible T such that $(A,H) \overset{T}{\mapsto} (J,G)$ where

$$J = \left[\begin{array}{cc|cc|cc} 2 & 1 & & & & \\ 0 & 2 & & & & \\ \hline & & \lambda & 1 & & \\ & & 0 & \lambda & & \\ \hline & & & & \overline{\lambda} & 1 \\ & & & & 0 & \overline{\lambda} \end{array}\right]$$

and $T = [T_1|T_2] = [\underbrace{g_0|g_1}_{T_1}|\underbrace{h_0|h_1|\gamma\overline{h_0}|\gamma\overline{h_1}}_{T_2}]$ for $\gamma \neq 0$.

- If A is real, then $(A-\lambda I)h_0 = 0 \Rightarrow (A-\overline{\lambda} I)\overline{h_0} = 0$

$$(A-\lambda I)h_1 = h_0 \Rightarrow (A-\overline{\lambda} I)\overline{h_1} = \overline{h_0}.$$

Jordan bases, having the above property, are called γ-Conjugate Symmetric Basis (γ-CS).

FOSC basis

FO basis	γ-CS basis
$(A, H) \overset{S}{\mapsto} (J, P)$	$(A, H) \overset{T}{\mapsto} (J, G)$
$S = [f_0 \vert f_1 \vert g_0 \vert g_1 \vert h_0 \vert h_1]$	$T = [g_0 \vert g_1 \vert h_0 \vert h_1 \vert \gamma\overline{h_0} \vert \gamma\overline{h_1}]$
P is a sip matrix	G is not necessarily simple

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$S=[f_0\|f_1\|g_0\|g_1\|h_0\|h_1]$	$T=[g_0\|g_1\|h_0\|h_1\|\gamma\overline{h_0}\|\gamma\overline{h_1}]$
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Are there γ-*FOCS* bases? **YES**

New Weierstrass Type Form

This is a **NEW** canonical form for real matrices. The Weierstrass form claims that there is an FO basis and we showed that in addition it can be CS as well.

Existence of a γ-FOCS basis

Theorem (2019, S. Dogruer Akgul, A. M., V. Olshevsky)

Let $A \in \mathbb{C}^{n\times n}$ be a fixed H-selfadjoint matrix for some self-adjoint matrix H and J be its Jordan form. Assume, in addition, that both matrices A and H are real. Then N in $(A,H) \overset{N}{\mapsto} (J,P)$ can be chosen such that if we partition it in the agreement with

$$N = [N_1|\dots|N_\alpha|N_{\alpha+1}|\dots|N_\beta]$$

then each N_k has the form

$$N_k = [L_k|\gamma\overline{L}_k]$$

for some matrices L_k for $k = \alpha+1,\dots,\beta$.

Summary of what was explained so far

- We introduced a new concept of $FOCS$ basis.
- We proved if A is H-selfadjoint and A, H are real, then (A, H) has a γ-$FOCS$ basis.
- Is it Lipschitz stable? **YES**

Lipschitz stability of γ-FOCS basis

Theorem (2019, S. Dogruer Akgul, A. M., V. Olshevsky)

Let $A_0 \in \mathbb{R}^{n\times n}$ be a fixed H_0-selfadjoint matrix, where $H_0 \in \mathbb{R}^{n\times n}$, and let the columns of this matrix capture the FOCS-basis, that is, $(A_0, H_0) \overset{T_0}{\mapsto} (J, P)$. There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds for any H-selfadjoint matrix A with the pair (A, H) real and A invertible and similar to A_0, and

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there is an invertible matrix $N \in \mathbb{R}^{n\times n}$, whose columns capture the γ-FOCS basis of A, i.e. $(A, H) \overset{N}{\mapsto} (J, P)$, and such that

$$\|N - T_0\| \le K(\|A - A_0\| + \|H - H_0\|).$$

Rodman's Question

Recall: Leiba Rodman asked the following question:

Are there similar results for a well-known

Real Canonical (RC) form?

Stability

Theorem (2019, S. Dogruer Akgul, A. M., V. Olshevsky)

Let $A_0 \in \mathbb{R}^{n\times n}$ be a fixed H_0-selfadjoint matrix where $H_0^T = H_0 \in \mathbb{R}^{n\times n}$ is an invertible matrix. Let

$$(A_0, H_0) \overset{R_0}{\mapsto} (J_R, P)$$

be the mapping with the columns of R_0 capture the RC basis of A_0. There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any real pair (A, H) such that A is H-selfadjoint and A is similar to A_0 and

$$\|A - A_0\| + \|H - H_0\| < \delta.$$

Then, there exists a matrix R such that $(A, H) \overset{R}{\mapsto} (J_R, P)$, whose columns capture the RC basis of A, and such that

$$\|R - R_0\| \leq K(\|A - A_0\| + \|H - H_0\|).$$

Summary of results discussed so far

- Existence of a new canonical Jordan basis.
- Lipschitz stability of a γ-$FOCS$ basis.
- Lipschitz stability of a real canonical basis.

References

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THANK YOU!