Deciding Non-Compressible Blocks in Sparse Direct Solvers using Incomplete Factorization

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Context: Sparse direct solvers

Current status of sparse direct solvers on 3D problems

- $\Theta(n^2)$ time complexity
- $\Theta(n^{\frac{4}{3}})$ memory complexity

Low-rank representations to reduce these costs

- Different representations:
  - Non-hierarchical format like BLR
  - Hierarchical format like $\mathcal{H}$, HODLR, HSS, $\mathcal{H}^2$
- Recently introduced in many solvers:
  - MUMPS, PASTIX, STRUMPACK and many others

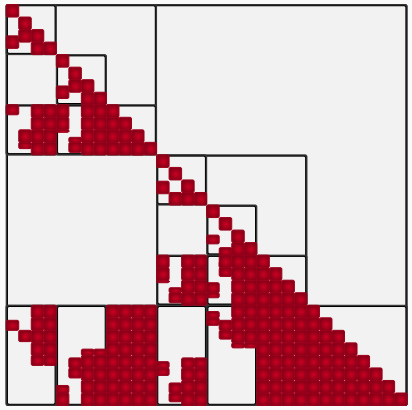
Sparse supernodal direct solver PaStiX

PaStiX 6.2.1 `http://gitlab.inria.fr/solverstack/pastix`

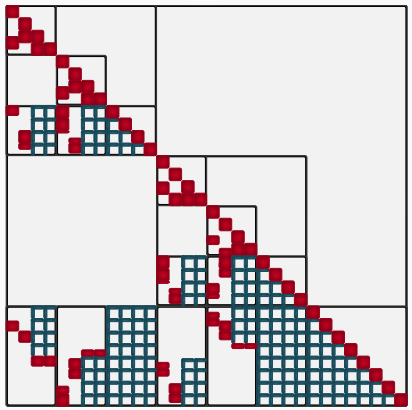
- Support shared/distributed memory with different schedulers:
  - Sequential
  - Static/Dynamic
  - PaRSEC/StarPU runtime systems with GPU support
- Cholesky, *LDL* and *LU* factorizations
- GMRES, CG, BiCG iterative refinements
- Wrappers:
  - F90/Python/Julia
  - PETSc (ongoing work)
- Non-hierarchical representation: block low-rank (BLR)

Block Low-Rank (BLR) representation (in PaStiX)

Full rank:



Block low-rank:



Approach

- Diagonal blocks are kept dense
- Off-diagonal blocks are compressed wrt an admissibility criterion (block size)

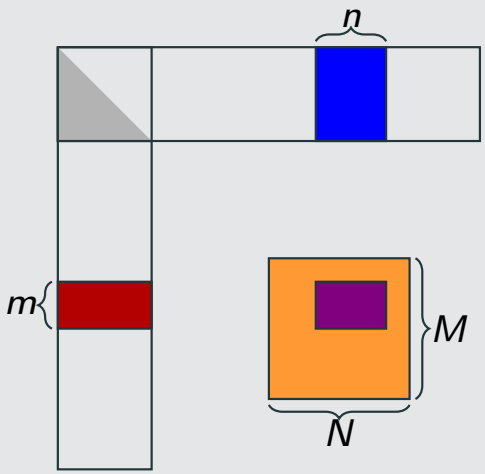
Operations

- TRSM are always performed on the **low-rank** form
- Two scenarios to apply the GEMM updates

Strategies to apply the updates ($C = C - AB$)

Cost of applying a $m \times n$ update to a $M \times N$ block.

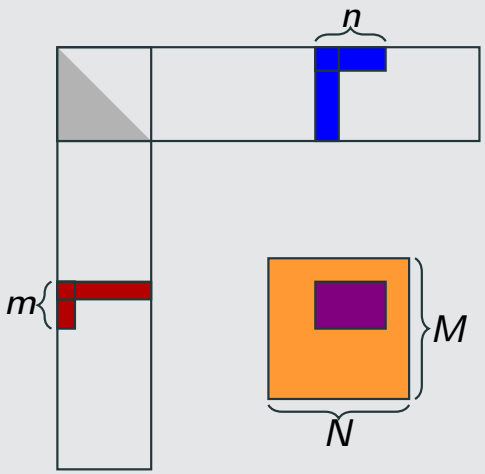
Full rank



- A single GEMM kernel
- Cost depends on mn

Non-fully structured

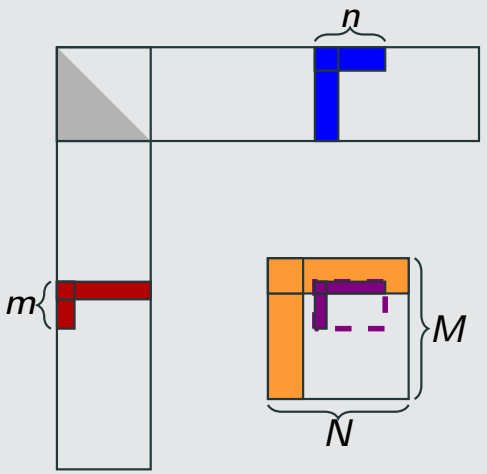
Just-In-Time



- Small GEMM kernels
- Cost depends on mn

Fully structured

Minimal Memory



- Complex operation with padding
- Cost depends on MN
- Recompression (RRQR or SVD) is needed

The two compression scenarios

Algorithm 1 Minimal Memory (MM) - Just in Time (JIT) Scenarios

```
1: for each off-diagonal block A_ij in A do
2:     Compress(A_ij)                          /* Compress if admissible */
3: for k=1 to N_cblk do
4:     Factorize
5:     for each off-diagonal block A_ij in cblk do
6:         Compress(A_ij)                      /* Compress if admissible */
7:     Solve
8:     Update
```

Just in Time (JIT)

- Late Compression
  - *After all Write*
  - *Before all Read tasks*
- Non fully structured updates
- ✗ Do not reduce the memory peak
- ✓ Reduce the flop count
- ✓ Reduce the time to solution

Minimal Memory (MM)

- Early Compression
  - *Before any other operation*
- Fully structured updates
- ✓ Reduce memory footprint
- ✗ May increase the flop count
- ✗ May increase the time to solution

Objective

Take advantage of JIT and MM at the same time

- Early compression of highly compressible blocks to reduce memory
- Late compression of poorly compressible blocks to reduce flop/time

How to determine the compressibility of the blocks?

- Marchal et al. [1] introduced models to determine the block compressibility at run-time.

Our approach

Exploit the fill-in levels concept of Incomplete LU (ILU) factorization

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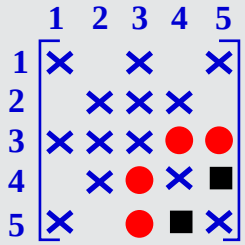
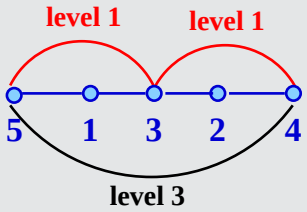
[1] L. Marchal, T. Marette, G. Pichon, and F. Vivien. Trading performance for memory in sparse direct solvers using low-rank compression. *Future Generation Computer Systems*, 130:307-320, 2022.

Block ILU Factorization

Recalls on ILU





- The fill-in level of an entry (in L) somehow illustrates the importance of its *magnitude*.
- Higher the fill-in level of a block, higher the distance of the interactions it represent in L.
- Easy and cheap heuristic to apply in a block-wise manner.

Low level blocks

- Have larger ranks
- Marginally improve memory footprint

$\rightarrow$ Suitable for late compression

High level blocks

- Have smaller ranks
- Greatly improve memory footprint

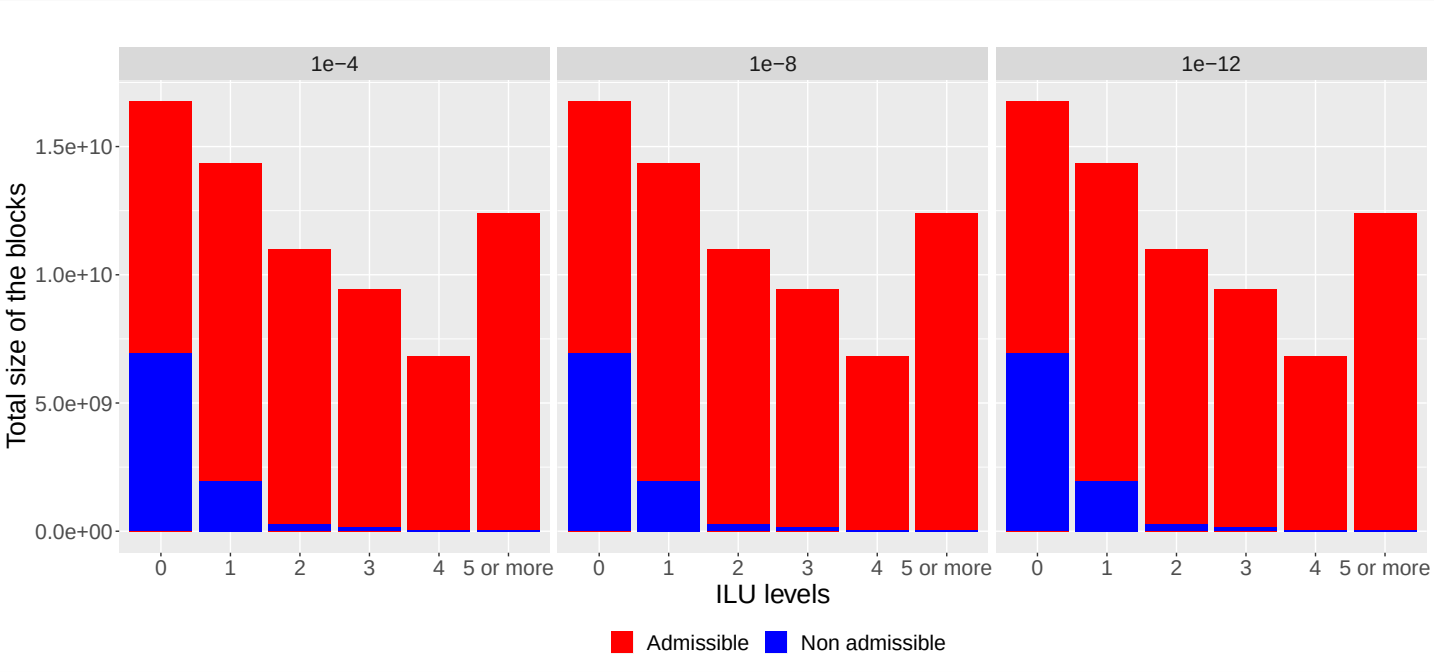
$\rightarrow$ Suitable for early compression

Cholesky based block ILU level definition

Algorithm 2 Block-wise symbolic ILU(*maxlevel*) factorization

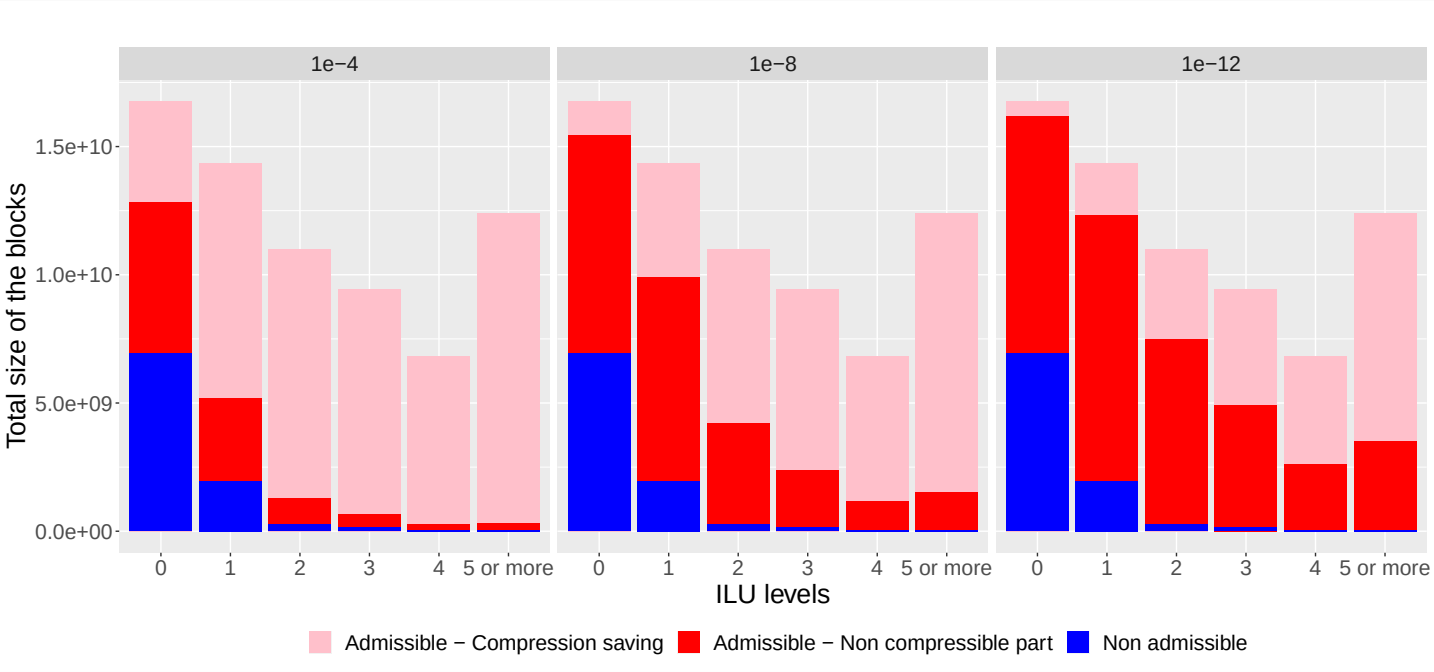
1: /* *Initialize the levels* */
2: **for all** block A_{ij} in A **do**
3: **if** $A_{ij} \neq 0$ **then**
4: $lvl(A_{ij}) = 0$
5: **else**
6: $lvl(A_{ij}) = \infty$
7: /* *Simulate the block factorization* */
8: **for all** column block A_{*k} in A **do**
9: **for all** block A_{ik} in A_{*k} **do**
10: **for all** block A_{jk} in A_{*k} (with $j > i$) **do**
11: $lvl(A_{ij}) = min(lvl(A_{ij}),\ lvl(A_{ik}) + lvl(A_{jk}) + 1)$

Admissibility of blocks at each ILU block level and tolerance



- Statistics on a set of 31 matrices from SuiteSparse Matrix Collection
- Red represents the memory used by the admissible blocks
- Blue represents the memory used by the non-admissible blocks

Compression ratio wrt ILU block levels and tolerance



- Statistics on a set of 31 matrices from SuiteSparse Matrix Collection
- Red shows the memory used by the compressed admissible blocks
- Pink shows the saved memory by compressing admissible blocks

- The higher the level, the higher the memory saving
- For a low tolerance criterion, good memory savings can be obtained considering only small levels of fill

BLR algorithm with the new ILU(*maxlevel*) heuristic

Algorithm 3 Minimal Memory - Just in Time - ILU(*maxlevel*) Scenarios

```
1:  for each off-diagonal block A_ij in A do
2:      if level(A_ij) > maxlevel then
3:          Compress(A_ij)                    /* Compress if admissible */
4:  for k=1 to N_cblk do
5:      Factorize
6:      for each off-diagonal block A_ij in cblk do
7:          if level(A_ij) <= maxlevel then
8:              Compress(A_ij)                /* Compress if admissible */
9:      Solve
10:     Update
```

maxlevel defines the criterion to switch from early to late compression according to the input matrix and tolerance

Experiments

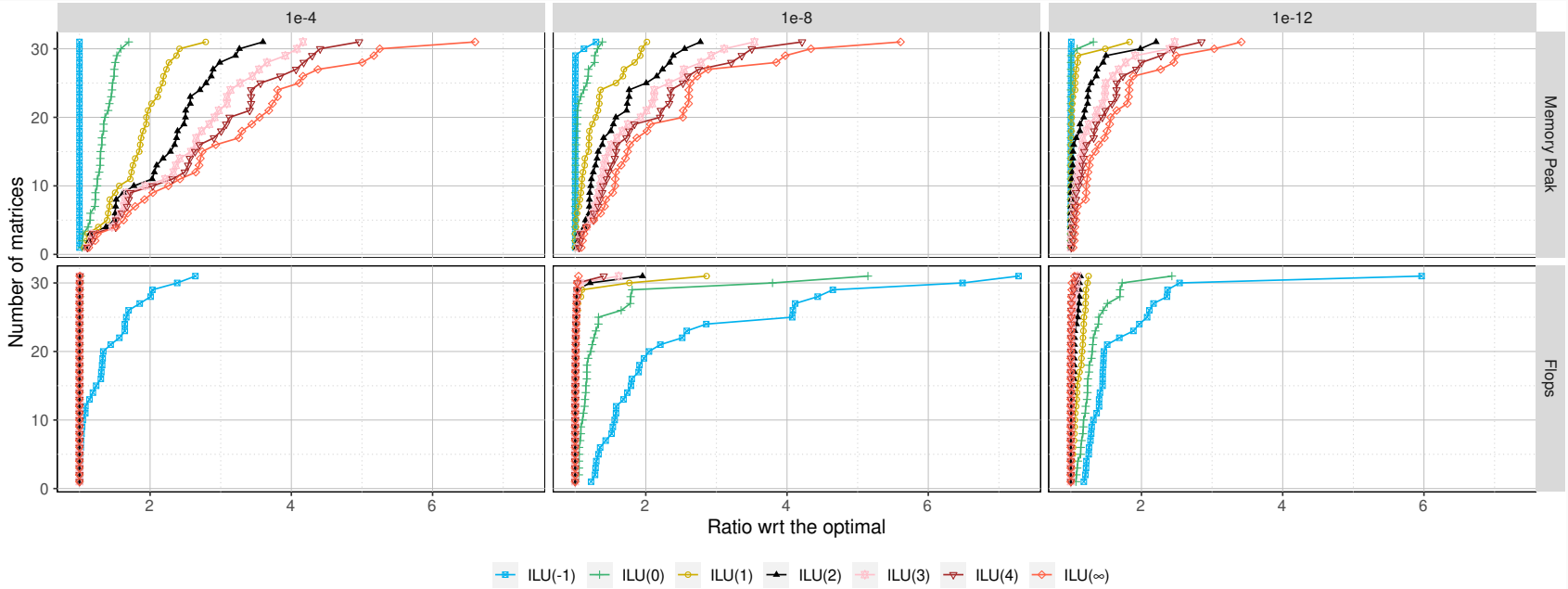
Context

- On a set of 31 matrices from the SuiteSparse Matrix Collection
- On a two INTEL Xeon E5-2680v3 12-cores running at 2.50 GHz and 128 GB of memory

Presentation of results

- We compare our heuristic with different *maxlevel* values ($ILU(maxlevel)$)
- $ILU(-1)$ stands for the existing *Minimal Memory* scenario
- $ILU(\infty)$ stands for the existing *Just-In-Time* scenario
- If a curve is closer to $x = 1$, this method is better on average

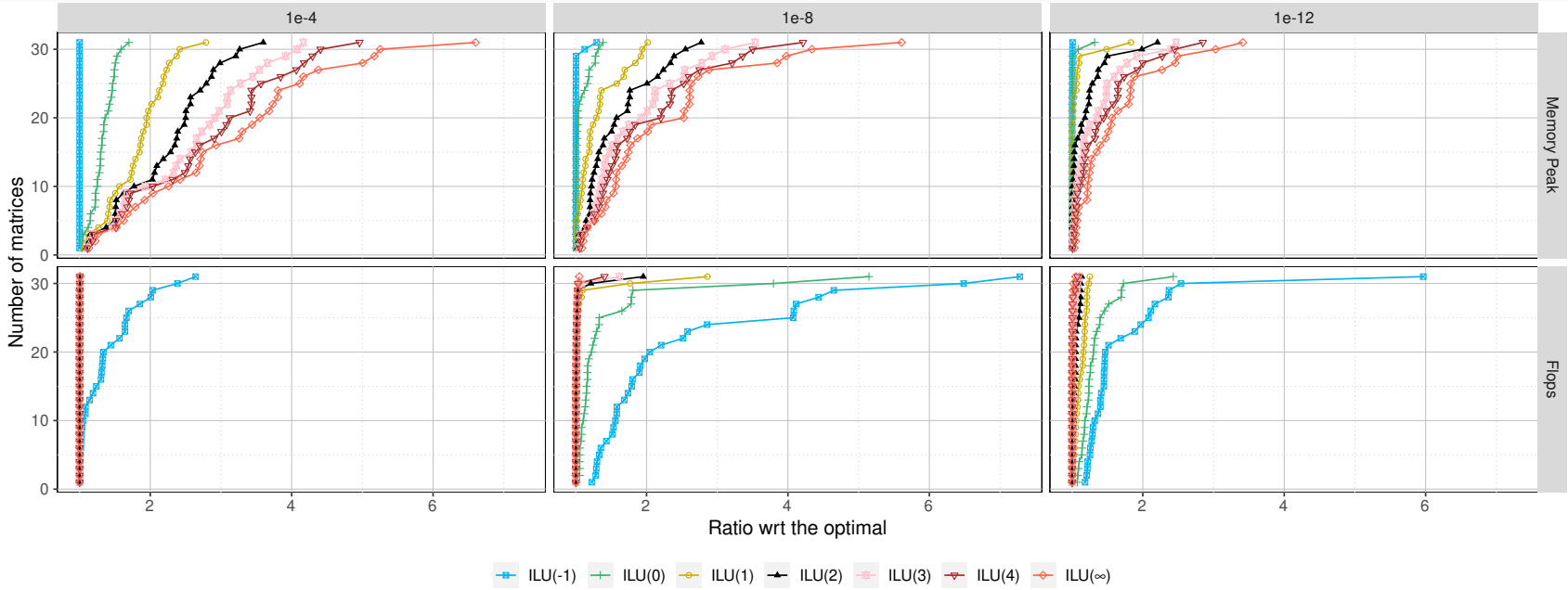
Impact of the ILU(k) Heuristic on the memory and flops



Memory consumption

- *Minimal Memory* has smallest memory, while *Just-In-Time* is worst
- The lower the level, the higher the memory consumption
- *ILU*(1) reaches a good compromise in average

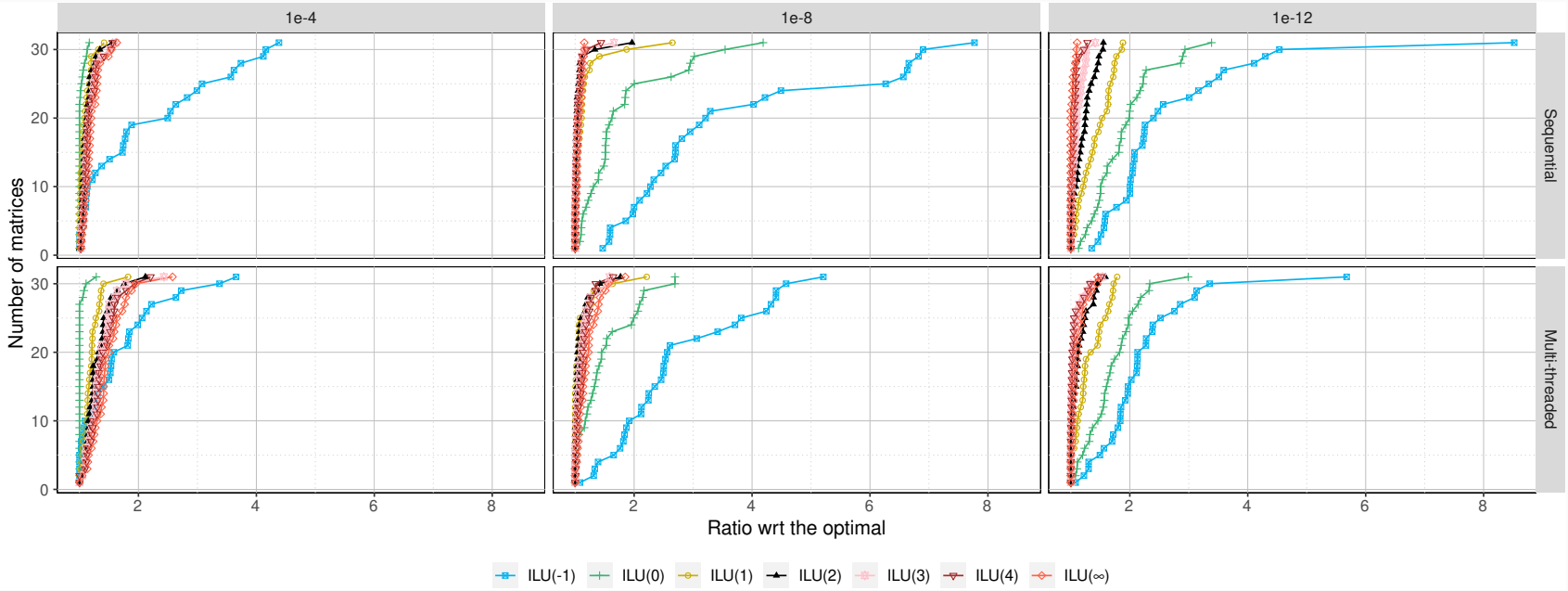
Impact of the ILU(k) Heuristic on the memory and flops



Flops

- *Just-In-Time* has smallest flops count, while *Minimal Memory* is worst
- The lower the level, the lower the flops count
- In many cases, the new heuristic is almost mingled with *Just-In-Time*

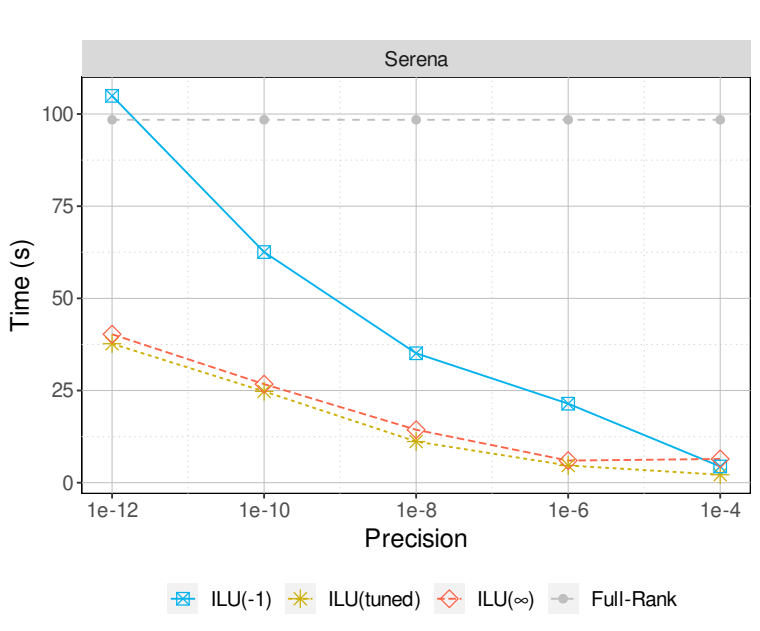
Impact of the ILU(k) Heuristic on the factorization time



Sequential vs multi-threaded (24 threads)

- The trend on the factorization time follows the trend on flops
- The ILU(k) heuristic behaves better than the *Just-In-Time* strategy in parallel: its execution time is often lower, especially for low tolerance

Tolerance scalability on the Serena Matrix



Tolerance	$ILU(tuned)$
$1e^{-4}$	$ILU(0)$
$1e^{-6}$	$ILU(2)$
$1e^{-8}$	$ILU(2)$
$1e^{-10}$	$ILU(4)$
$1e^{-12}$	$ILU(4)$

Table 1: Corresponding levels for $ILU(tuned)$ at each precision.

Observations

- By tuning correctly the level of fill for each tolerance, we can always outperform both the *Minimal Memory* and the *Just-In-Time* strategies
- Recall that when $ILU(k)$ is faster than *Just-In-Time*, it also reduces significantly its memory consumption

Conclusion

The ILU(k) heuristic

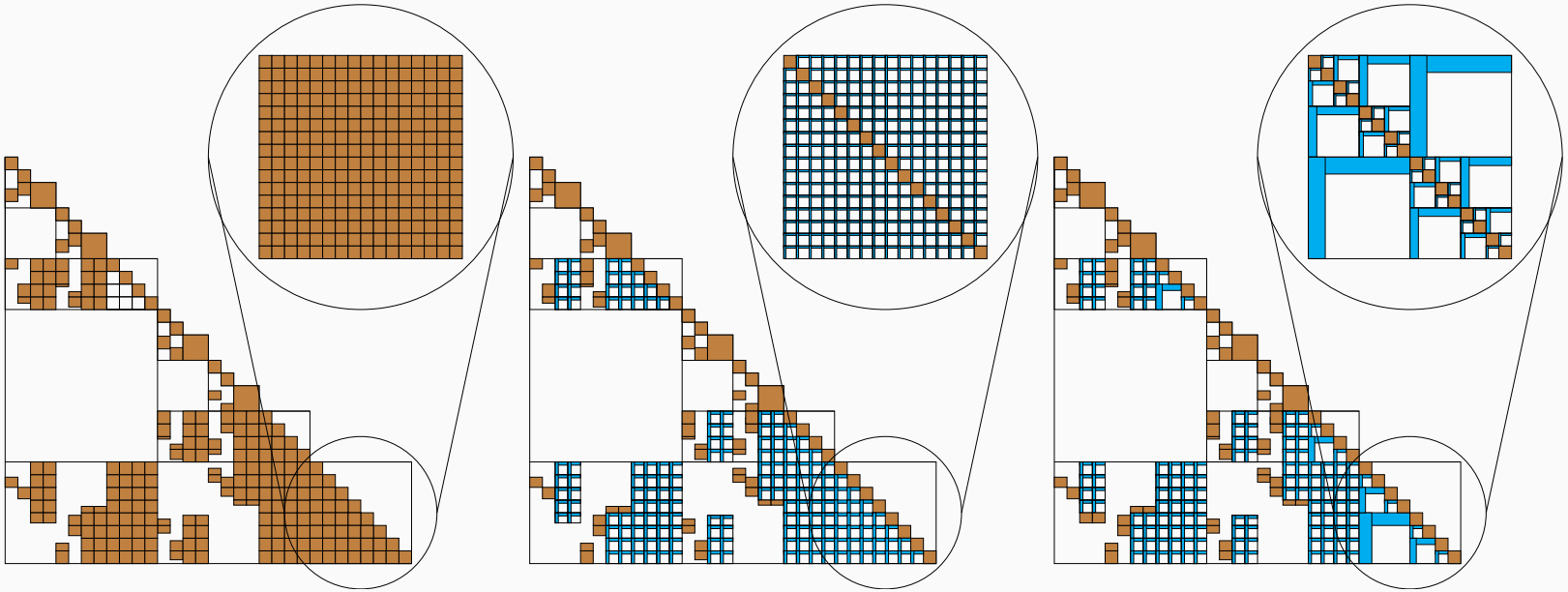
- Allows to reach good trade-offs between the *Minimal Memory* and the *Just-In-Time* strategies
- The new heuristic in sequential provides huge speedup and reduces flops with only a slight memory increase
- The new heuristic in parallel is even faster than *Just-In-Time*, with improved memory

Future work

- Infer automatically the level depending on the user's requirements and matrices
- Inject ILU based information to the run-time preselection method offered by Marchal et al.
- Reduce cost by recompressing accumulated updates

Thank You!

Low-rank representations



- Full-rank format: Too much time and memory
- Low-rank approximation ($A \approx U \times V^T$) on the blocks of:
  - Non-hierarchical Format: Simpler to implement and more flexible in parallel environments
  - Hierarchical Format: Less computational complexity and memory consumption