The effect of approximate coarsest-level solves on the convergence of multigrid V-cycle methods

Petr Vacek, Erin Carson, Kirk M. Soodhalter

Charles University in Prague

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FACULTY OF MATHEMATICS AND PHYSICS
Charles University

inEXASCALE

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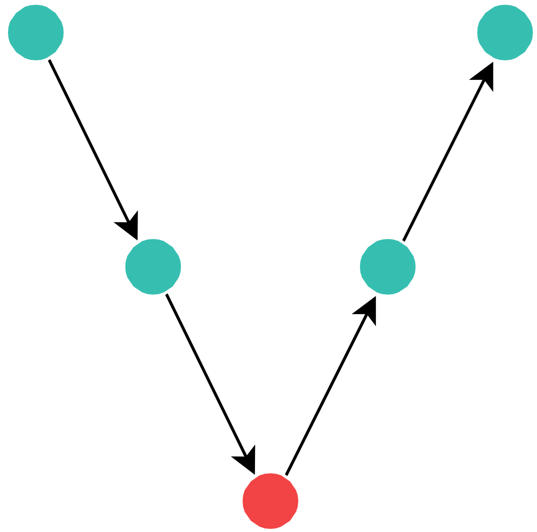
Introduction

Find x : $Ax = b$.

x^{prev} x^{new}



Find v_0: $A_0 v_0 = f_0$.

- smoothing
- solving on the coarsest level

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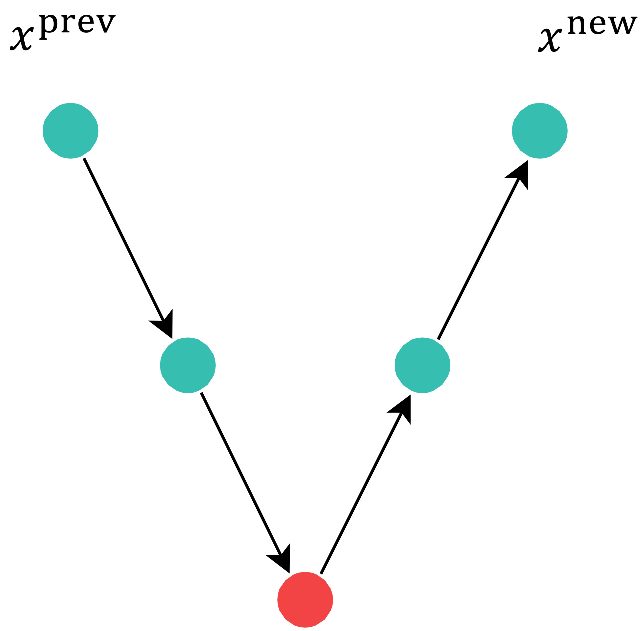

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<u>smoothing</u>

<u>solving on the coarsest level</u>

direct solver based on LU decomposition

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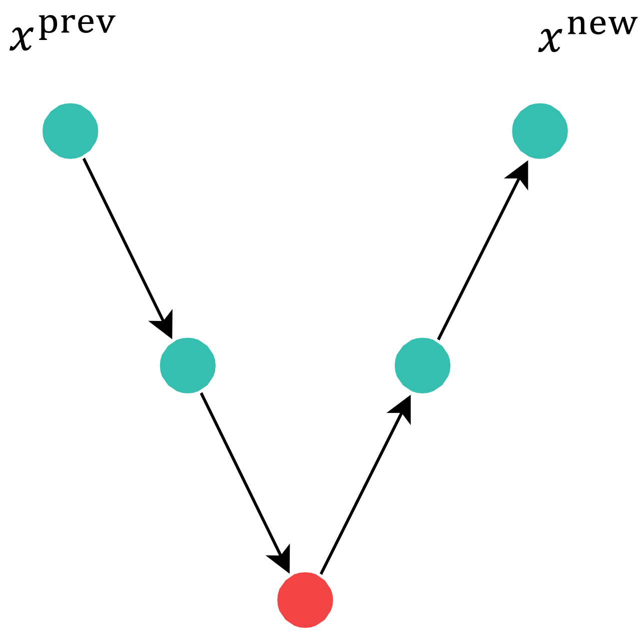

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$v_{0,\text{in}} \approx v_0$

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direct solver based on LU decomposition

Krylov subspace methods

Block Low Rank direct solver

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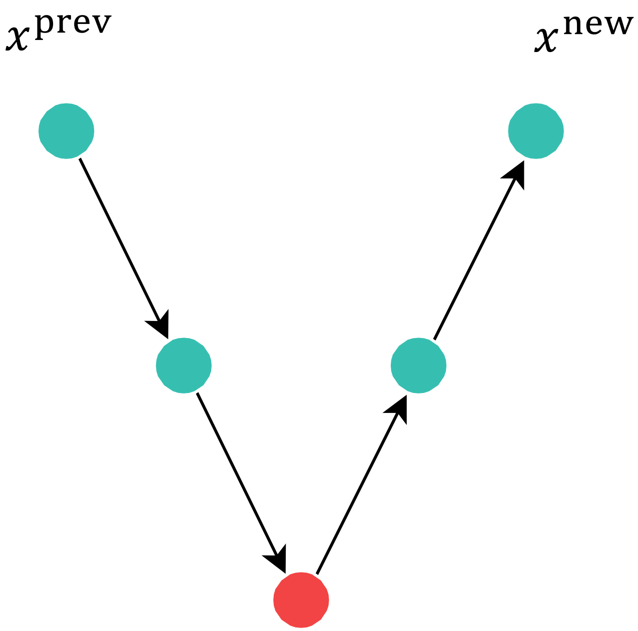

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smoothing

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direct solver based on LU decomposition

Krylov subspace methods

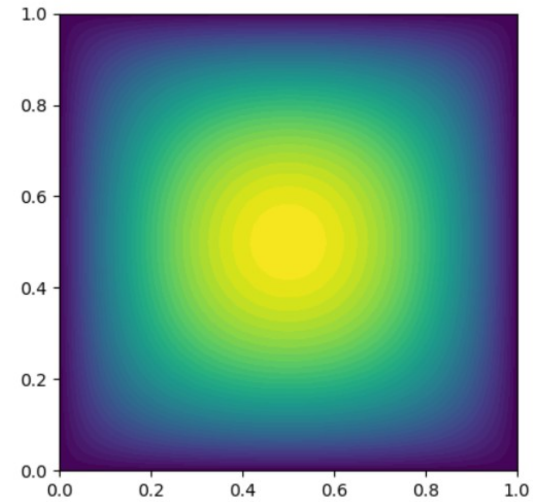
$$\frac{\|f_0 - A_0 v_{0,\mathrm{in}}\|}{\|f_0\|} \leq \tau$$

Block Low Rank direct solver

Motivational experiment

<u>Poisson problem</u>

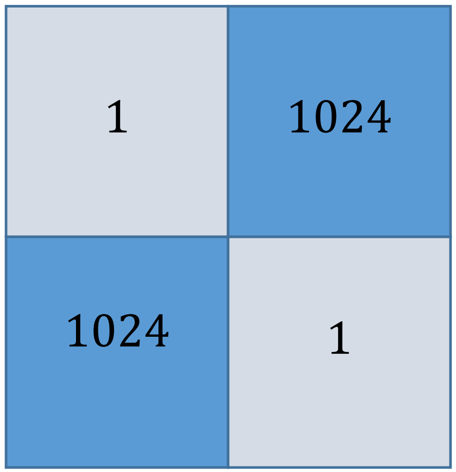
$$-\Delta u = 1 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$



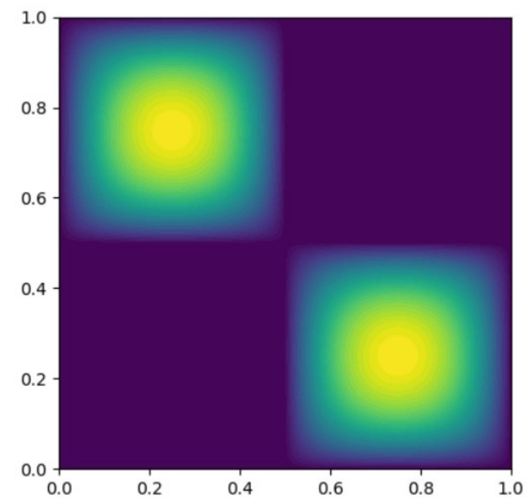
computed solution

<u>jump-1024 problem</u>

$$-\nabla \cdot (k(x)\nabla u) = 1 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$



coefficient $k(x)$



computed solution

Motivational experiment

Discretization

Galerkin FEM – continuous piecewise linear functions

uniform refinement

6 levels

coarsest-level: 1.52E+03 DoF

finest level: 1.64E+06 DoF

V-cycle method V(1,1)

smoother: symmetric Gauss-Seidel method

solver: MATLAB backslash or CG

zero initial approximation

$\left\| x - x^{(n)} \right\|_A \leq \theta, \quad \theta = 10^{-4}$ or $\theta = 10^{-11}$

Motivational numerical experiment

	Poisson problem				jump-1024 problem			
	condition number of the coarsest-level matrix							
	6.48E+02				1.66E+05			
	finest level tolerance θ				finest level tolerance θ			
	1.00E-04		1.00E-11		1.00E-04		1.00E-11	
τ	V-cycles	total CG it.						
5.00E-01	5	68						
2.50E-01	3	62						
1.25E-01	3	71						
6.25E-02	2	63						
3.13E-02	2	71						
1.56E-02	2	79						
7.81E-03	2	88						
3.91E-03	2	96						
1.95E-03	2	100						
9.77E-04	2	106						
4.88E-04	2	110						
2.44E-04	2	115						
1.22E-04	2	120						
6.10E-05	2	125						
3.05E-05	2	129						
1.53E-05	2	132						
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A\b	2							

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Can we define coarsest-level stopping criteria that would yield a computed V-cycle approximation „close“ to the V-cycle approximation which would be obtained by solving the coarsest-level problems exactly?

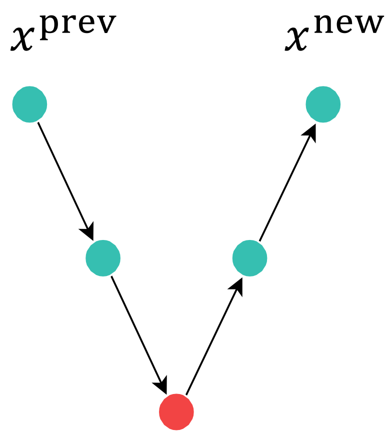
Setting

A - symmetric positive definite matrix

V-cycle method, $R_j = P_j^T$, $A_j = P_j^T A P_j$, Galerkin condition

V-cycle with exact solver converges

Find x : $Ax = b$.



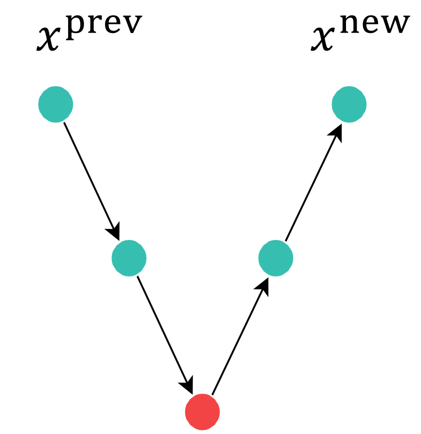
Find v_0: $A_0 v_0 = f_0$.

$v_{0,\text{in}} \approx v_0$

Analysis

$$x - x_{\text{in}}^{\text{new}} = x - x_{\text{ex}}^{\text{new}} + x_{\text{ex}}^{\text{new}} - x_{\text{in}}^{\text{new}}$$

Find x : $\quad Ax = b$.



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Analysis

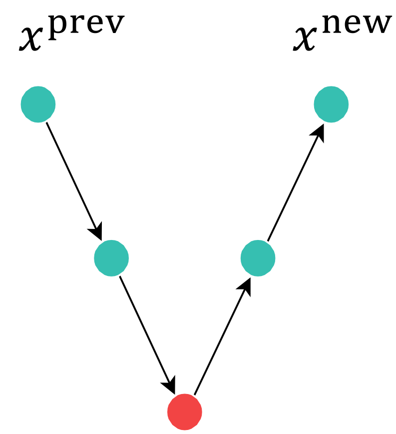
$$x - x_{\text{in}}^{\text{new}} = x - x_{\text{ex}}^{\text{new}} + x_{\text{ex}}^{\text{new}} - x_{\text{in}}^{\text{new}}$$
$$= E(x - x^{\text{prev}}) + S(v_0 - v_{0,\text{in}})$$

error propagation matrix of V-cycle with exact solver

error of previous approximation

error on the coarsest-level

Find x : $Ax = b$.



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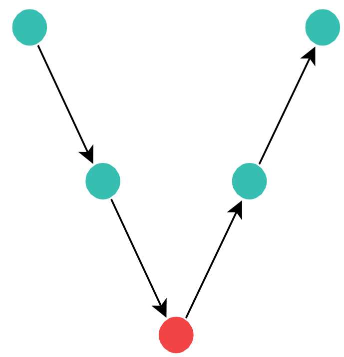
$$\|x - x_{\text{in}}^{\text{new}}\|_A \le \|E\|_A \cdot \|x - x^{\text{prev}}\|_A + \underbrace{\|S\|_{A_0,A}}_{\le 1} \cdot \left\|v_0 - v_{0,\text{in}}\right\|_{A_0}$$

$$\|S\|_{A_0,A} := \max_{v \in \mathbb{R}^{n_0}, v \ne 0} \frac{\|Sv\|_A}{\|v\|_{A_0}}$$

Find x : $Ax = b$.

x^{prev} x^{new}



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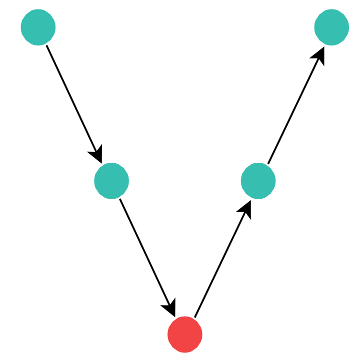
x^{prev} x^{new}



Find v_0: $A_0 v_0 = f_0$.

$v_{0,\text{in}} \approx v_0$

<u>Relative coarsest-level accuracy assumption</u>

Let $\gamma > 0$: $\left\|v_0 - v_{0,\text{in}}\right\|_{A_0} \leq \gamma \cdot \|x - x^{\text{prev}}\|_A$

Result:

$$\|x - x_{\text{in}}^{\text{new}}\|_A \leq \left(\|E\|_A + \gamma \cdot \|S\|_{A_0,A}\right)\|x - x^{\text{prev}}\|_A$$

Coarsest-level solver with relative residual tolerance

$$\frac{\left\|f_0 - A_0 v_{0,\mathrm{in}}\right\|}{\|f_0\|} \leq \tau$$

$$\|x - x_{\mathrm{in}}^{\mathrm{new}}\|_A \leq \left(\|E\|_A + \tau \cdot \|T\| \cdot \|S\|_{A_0,A} \cdot \sqrt{\|A\| \cdot \|A_0^{-1}\|} \right) \|x - x^{\mathrm{prev}}\|_A$$

Analysis

$$x - x_{\text{in}}^{\text{new}} = x - x_{\text{ex}}^{\text{new}} + x_{\text{ex}}^{\text{new}} - x_{\text{in}}^{\text{new}}$$

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error propagation matrix of V-cycle with exact solver

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$$\|x - x_{\text{in}}^{\text{new}}\|_A \leq \|E\|_A \cdot \|x - x^{\text{prev}}\|_A + \underbrace{\|S\|_{A_0,A}}_{\leq 1} \cdot \left\|v_0 - v_{0,\text{in}}\right\|_{A_0}$$

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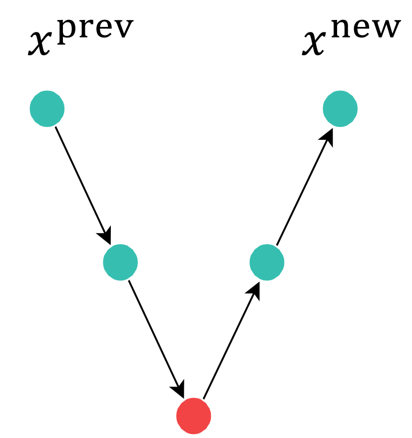

Find v_0: $A_0 v_0 = f_0$.

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<u>Absolute coarsest-level accuracy assumption</u>

Let $\epsilon > 0$: $\left\|v_0 - v_{0,\text{in}}\right\|_{A_0} \leq \epsilon$

Result:

$$\left\|x_{\text{ex}}^{(n)} - x_{\text{in}}^{(n)}\right\|_A \leq \frac{\epsilon \cdot \|S\|_{A_0,A}}{1 - \|E\|_A} \leq \frac{\epsilon}{1 - \|E\|_A}$$

New coarsest-level stopping criterion

Can we define coarsest-level stopping criteria that would yield a computed V-cycle approximation „close" to the V-cycle approximation which would be obtained by solving the coarsest-level problems exactly?

Ingredients:

Upper bound on the error of the coarsest level solver $\left\|v_0 - v_{0,\mathrm{in}}\right\|_{A_0} \leq \eta\left(v_{0,\mathrm{in}}\right)$

Estimate of the convergence rate $\|E\|_A \leq \alpha$

Stop the coarsest-level solver when

$\frac{\eta\left(v_{0,\mathrm{in}}\right)}{(1-\alpha)} \leq \theta,$ where θ is a parameter chosen by the user.

Then $\left\|x_{\mathrm{ex}}^{(n)} - x_{\mathrm{in}}^{(n)}\right\|_A \leq \theta,$

$\left\|x - x_{\mathrm{in}}^{(n)}\right\|_A \leq \left\|x - x_{\mathrm{ex}}^{(n)}\right\|_A + \theta$

Numerical Experiments

Stop the coarsest-level solver when

$$\frac{\eta(v_{0,\mathrm{in}})}{(1-\alpha)} \leq \theta$$

$\|E\|_A < \alpha = 2/3$

$\theta = 10^{-4}$ or 10^{-11}

(GR) Gauss-Radau bound on A-norm of the error in CG [Meurant, Tichý2023]

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GR	2	82	9	674	2	743	9	6489

Conclusions

New approach for analyzing the effects of approximate coarsest-level solves on the convergence of multigrid V-cycle methods

New coarsest-level stopping strategy taylored to multigrid methods

Future work:

Algebraic multigrid methods

CG preconditioned by multigrid with inexact coarsest level solver

P. V., E. Carson and K. M. Soodhalter, The Effect of Approximate Coarsest-Level Solves on the Convergence of Multigrid V-Cycle Methods, SIAM Journal on Scientific Computing (accepted; in press), *https://arxiv.org/abs/2306.06182*

A posteriori error estimates based on multilevel decompositions with large problems on the coarsest level

Petr Vacek, Jan Papež and Zdeněk Strakoš

Multilevel a posteriori error estimate [Rüde 1993]

$-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

$\mathrm{A}x = b$, FEM P1 discretizarion

$\hat{x} \approx x$

$r_j = P_j^T (b - A\hat{x})$

$$\eta_J = \left(\sum_{j=1}^{J} r_j^T \operatorname{diag}\left(A_j\right)^{-1} r_j + r_0^T A_0^{-1} r_0 \right)^{\frac{1}{2}}$$

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$$c\eta_J \le \|x - \hat{x}\|_A \le C\eta_J$$

Multilevel a posteriori error estimate [Rüde 1993]

$-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

$\mathrm{A}x = b$, FEM P1 discretizarion

$\hat{x} \approx x$

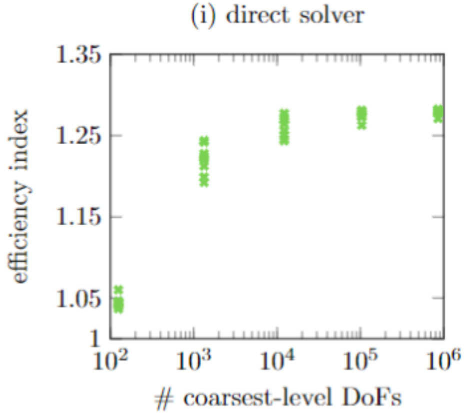
$r_j = P_j^T(b - A\hat{x})$

$$\eta_J = \left(\sum_{j=1}^{J} r_j^T \operatorname{diag}\left(A_j\right)^{-1} r_j + r_0^T A_0^{-1} r_0\right)^{\frac{1}{2}}$$

$$c\eta_J \le \|x - \hat{x}\|_A \le C\eta_J$$

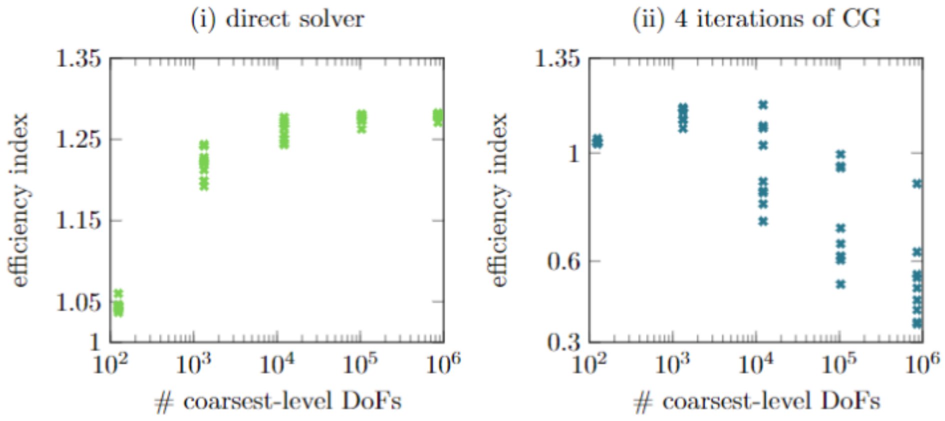
Numerical experiment

$$I = \frac{C\left(\sum_{j=1}^{J} r_j^T \operatorname{diag}(A_j)^{-1} r_j + r_0^T A_0^{-1} r_0\right)^{\frac{1}{2}}}{\|x - \hat{x}\|_A}$$



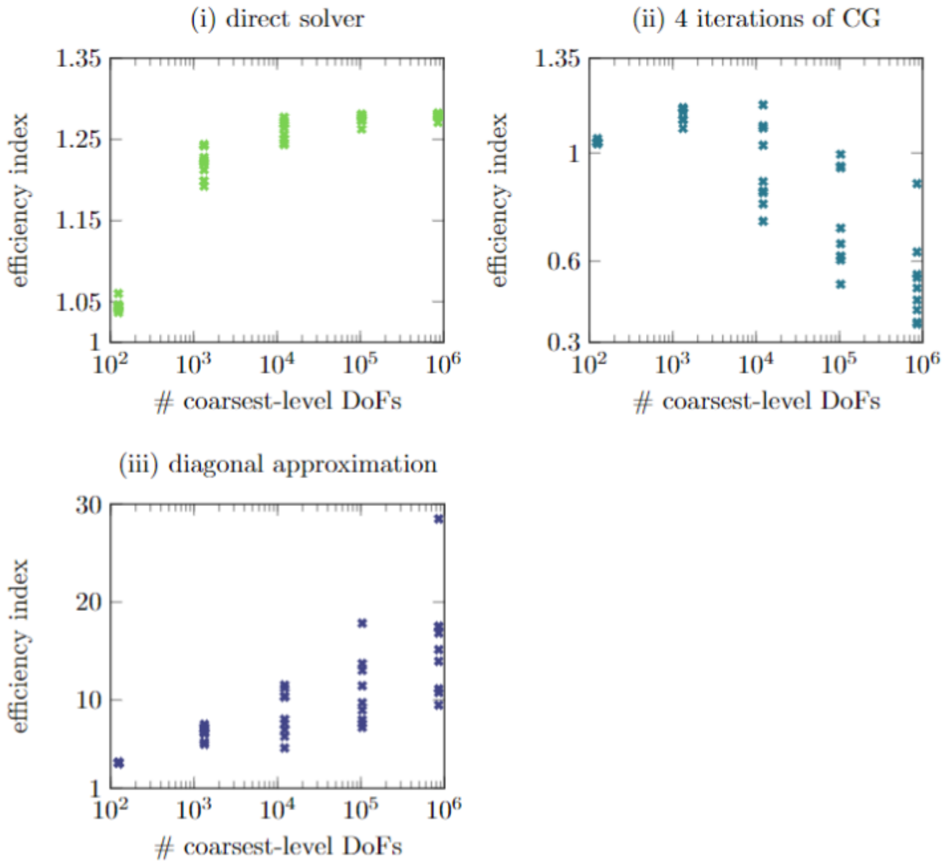
Numerical experiment

$$I = \frac{C\left(\sum_{j=1}^{J} r_j^T \operatorname{diag}(A_j)^{-1} r_j + r_0^T A_0^{-1} r_0\right)^{\frac{1}{2}}}{\|x - \hat{x}\|_A}$$



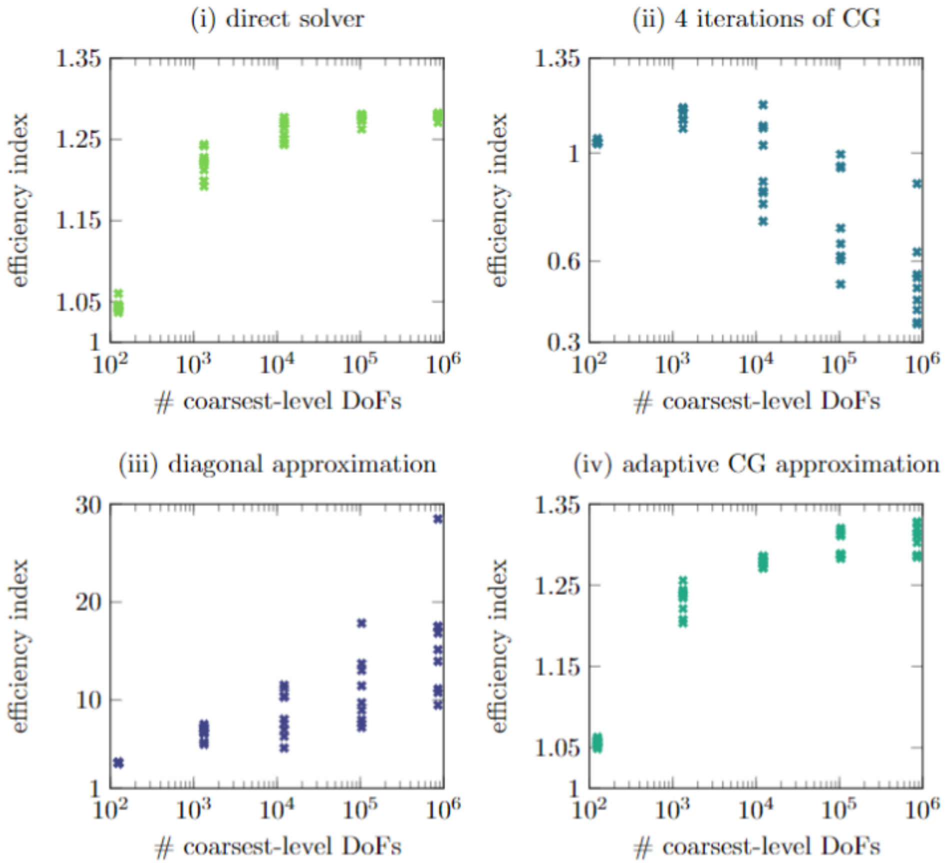
Numerical experiment

$$I = \frac{C\left(\sum_{j=1}^{J} r_j^T \operatorname{diag}(A_j)^{-1} r_j + r_0^T A_0^{-1} r_0\right)^{\frac{1}{2}}}{\|x - \hat{x}\|_A}$$



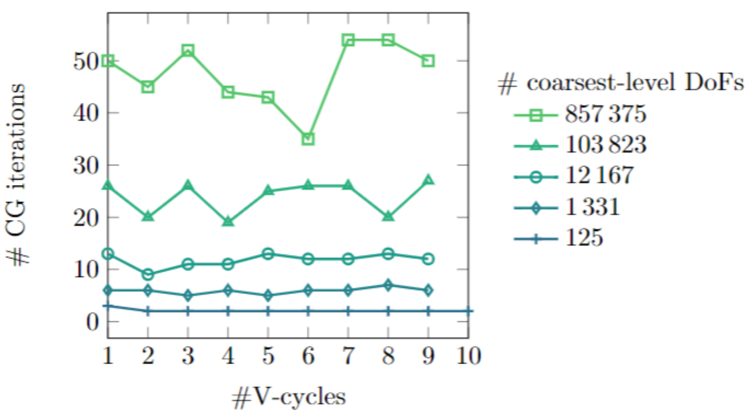
Numerical experiment

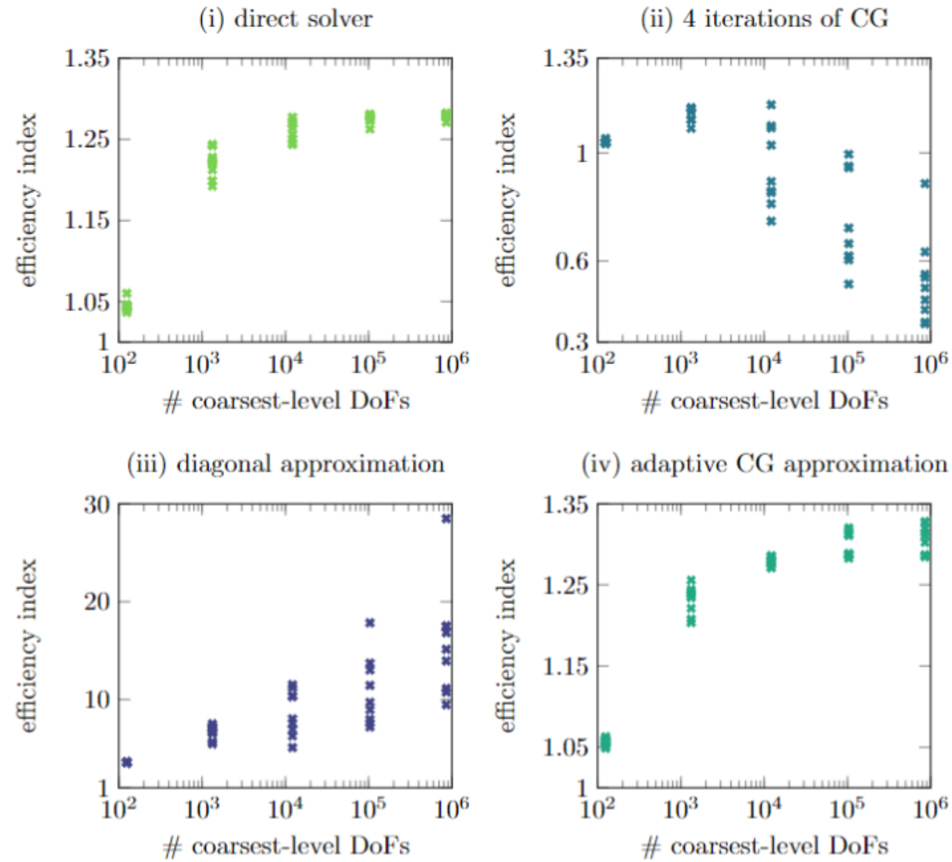
$$I = \frac{C\left(\sum_{j=1}^{J} r_j^T \operatorname{diag}(A_j)^{-1} r_j + r_0^T A_0^{-1} r_0\right)^{\frac{1}{2}}}{\|x - \hat{x}\|_A}$$



Numerical experiment

$$I = \frac{C\left(\sum_{j=1}^{J} r_j^T \operatorname{diag}(A_j)^{-1} r_j + r_0^T A_0^{-1} r_0\right)^{\frac{1}{2}}}{\|x - \hat{x}\|_A}$$





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