Preconditioning and iteration for indefinite linear systems

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$$\boldsymbol{Ax = b}, \qquad \boldsymbol{A^{-1}} \text{ exists}$$

Simple iteration: split $\boldsymbol{A = M - N}$ and for some $\boldsymbol{x_0}$

$$\textbf{solve} \quad \boldsymbol{Mx_{k+1} = Nx_k + b}, \quad \boldsymbol{k = 0, 1, \ldots} \qquad (\star)$$

$\boldsymbol{M}$: splitting matrix or preconditioner (invertible)

Convergence: for any $\boldsymbol{x_0}$, $\boldsymbol{\{x_k\}}$ converges to the solution $\Leftrightarrow |\boldsymbol{\lambda(M^{-1}N)}| = |\boldsymbol{\lambda(I - M^{-1}A)}| < \boldsymbol{1}$; contractive.

Equivalently $\boldsymbol{\lambda(M^{-1}A) \subset B(1, 1)}$, the open unit ball centred at $\boldsymbol{1}$.

If ever $\boldsymbol{M^{-1}A}$ has an eigenvalue with negative real part then ($\star$) certainly can not be contractive.

Polynomial iteration (Krylov, Chebyshev,...) require $\boldsymbol{p(0) = 1}$ ($\Rightarrow$ MINRES /GMRES)

The above applies to any matrix. Now consider A, M real symmetric:

If $A = A^T$: $inertia(A) = (p, n, z)$ where A has $p$ositive, $n$egative, $z$ero eigenvalues

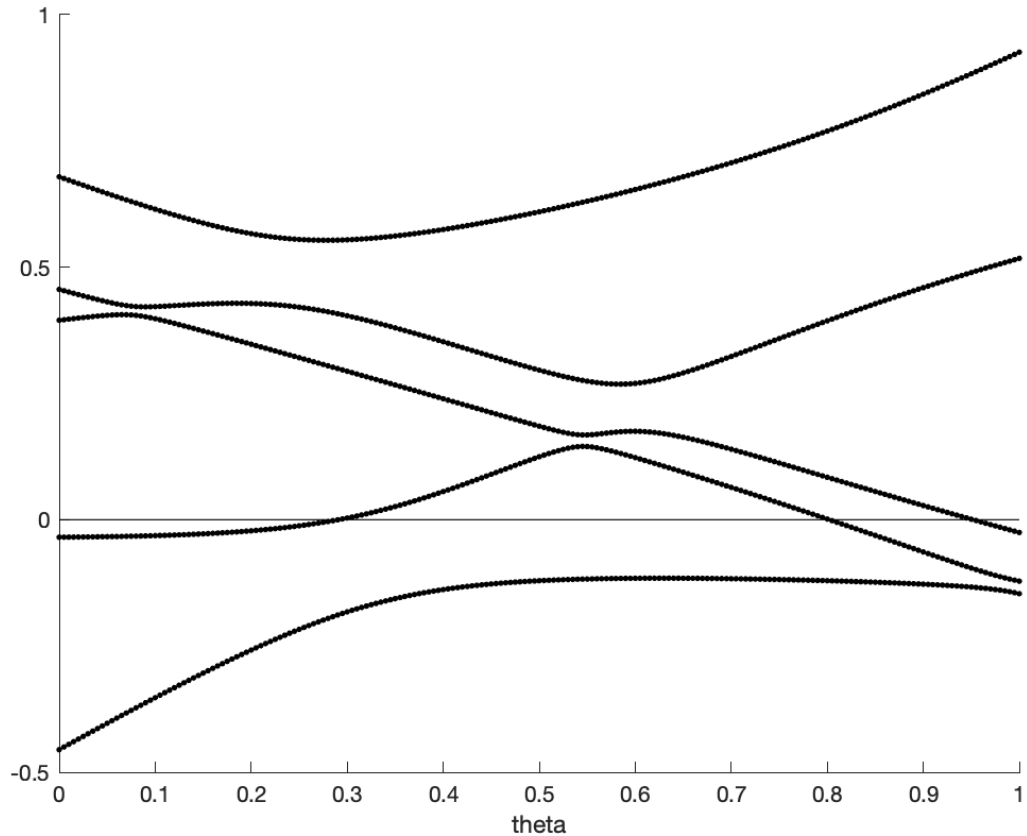
Lemma: if $inertia(A) \neq inertia(M)$ then $M^{-1}A$ has at least one real negative eigenvalue $\Rightarrow$ $(\star)$ not convergent

Proof
$T(\theta) = (1-\theta)A + \theta M$ is real symmetric $\Rightarrow$ real eigenvalues continuous in θ; $T(0) = A, T(1) = M$.
Different inertia $\Rightarrow$ there is $\widehat{\theta} \in (0, 1)$ with $T(\widehat{\theta})$ singular. That is

$$(1-\widehat{\theta})A + \widehat{\theta}M \quad \text{and so} \quad A - \widehat{\theta}/(\widehat{\theta}-1)M$$

is singular i.e. $\widehat{\theta}/(\widehat{\theta}-1) < 0$ is an eigenvalue of $M^{-1}A$. $\square$

Example



$$A = \begin{bmatrix} 0.33 & -.05 & -.29 & 0.01 & 0.01 \\ -.05 & 0.36 & -.11 & -.22 & -.19 \\ -.29 & -.11 & -.32 & 0.11 & -.01 \\ 0.01 & -.22 & 0.11 & 0.49 & -.12 \\ 0.01 & -.19 & -.01 & -.12 & 0.18 \end{bmatrix}, M = \begin{bmatrix} 0.14 & 0.10 & 0.25 & 0.09 & -.28 \\ 0.10 & -.07 & 0.02 & 0.08 & -.11 \\ 0.25 & 0.02 & 0.49 & -.11 & -.23 \\ 0.09 & 0.08 & -.11 & 0.24 & -.34 \\ -.28 & -.11 & -.23 & -.34 & 0.35 \end{bmatrix}$$

Examples:

- A SPD, usually M SPD *(Varga)*
- A SIND, M SPD (eg. block diagonal preconditioning of saddle point problems), inertia unchanged

$$M^{\frac{1}{2}}(M^{-1}A)M^{-\frac{1}{2}} = M^{-\frac{1}{2}}AM^{-\frac{1}{2}}$$

is symmetric and congruent to A so has the same inertia as A (Sylvester's Law of inertia)

- *constraint preconditioning* of saddle point systems

$$M = \begin{bmatrix} W & B^T \\ B & 0 \end{bmatrix}, A = \begin{bmatrix} H & B^T \\ B & 0 \end{bmatrix}$$

$inertia(A) = inertia(M)$ and $\lambda(M^{-1}A)$ all real, positive when W, H SPD *(Keller, Gould, W)*

Multigrid: *Braess-Sarazin*

Note $\boldsymbol{A}$ SIND, $\boldsymbol{M}$ SIND is generally difficult (eigenvalues of $\boldsymbol{M}^{-1}\boldsymbol{A}$ can be complex), but for saddle point problem

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{H} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix},$$

$\boldsymbol{H} \in \mathbb{R}^{n \times n}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$ inertia is $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{0})$ when $\boldsymbol{H}$ SPD

so eg. *Vanka* splitting for Stokes (Navier-Stokes?) can aim for this inertia

Note: Condition for contraction is necessary, not sufficient eg.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{-1} \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} \boldsymbol{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

Avoidance of Crossing

If $inertia(A) = (p, n, 0)$, $inertia(M) = (p + r, n - r, 0)$ $(-p \leq r \leq n)$, then almost certainly $M^{-1}A$ has $|r| + 2s$ real, negative eigenvalues for some $s \in \{0, 1, 2, \ldots, \lfloor \frac{p+n-r}{2} \rfloor\}$.

Proof

Eigenvalue avoidance $\Rightarrow$ trajectories $\lambda(\theta)$ of $T(\theta)$ do not generally intersect

$\Rightarrow$ $|r|$values $\widehat{\theta}_j, j = 1, \ldots, r$ where $T(\widehat{\theta}_j)$ is singular i.e.

$$\frac{\widehat{\theta}_j}{\widehat{\theta}_j - 1} < 0$$

must be an eigenvalue of $M^{-1}A$ for $j = 1, 2, \ldots, |r|$.

Trajectories can cross axis an even number of times ($\Rightarrow s$)$\square$

Positive real eigenvalues

$$S(\theta) = (1-\theta)A + \theta(-M),$$

$S(0) = A, S(1) = -M$ with

$inertia(A) = (p, n, 0), inertia(-M) = (n-r, p+r, 0)$

Almost certainly $M^{-1}A$ has $|p+r-n| + 2t$ real, positive eigenvalues for some $t \in \{0, 1, 2, \ldots, \min\left(\lfloor\frac{2p+r}{2}\rfloor, \lfloor\frac{2n-r}{2}\rfloor\right)\}$.

Conclusion

Indefinite preconditioning is generally a pretty tricky business!